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ON THE STRUCTURE OF THETA LIFTS OF DISCRETE SERIES FOR DUAL PAIRS $(Sp(n), O(V))$

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ABSTRACT

The purpose of this paper is to prove some fundamental results on the structure of theta lifts of discrete series.

Introduction

The purpose of the present paper is to prove some fundamental results on the structure of theta lifts of discrete series. They show that the lifts of discrete series behave very much like the lifts of supercuspidal representations (Théorème principal in [MVW, page 69]). Some of that is already well-known (see [M3], Theorem 4.1), and it is just a corollary (see Corollary 6.1) of more general and more precise results obtained here (Theorems 6.1, 6.2).

The proof of Theorem 4.1 in [M3] rely on the classification of discrete series for classical groups due to Mœglin and Tadić [Mœ, MT]. Although very elegant and useful for various computations, the classification of Mœglin and Tadić, is based on an assumption that is not verified yet to its full extent. Therefore, we do not use their classification, but some very simple properties of discrete series that we verify in Theorem 5.1. The ideas used in the proof of Theorem 5.1 are all contained in [Mœ, MT], but the proof of Theorem 5.1 is entirely based on Jacquet module technique of Bernstein–Zelevinsky and Tadić combined with some analytic results established in [W2].

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Let F be a nonarchimedean field of characteristic different than 2. We look at usual towers of even-orthogonal or symplectic groups $G_n = G(V_n)$, $n \geq n_0$. (See Section 1 for precise definition.) They are groups of isometries of F-spaces $(V_n, (\ , \ ))$, where $2n = \dim V_n$ and the form $(\ , \ )$ is nondegenerate and it is skew-symmetric if the tower is symplectic and symmetric otherwise built up from an anisotropic space V_{n_0}, $n_0 = 0, 1, 2$, adding $n - n_0$-hyperbolic planes. We fix one more tower of groups $G'_m = G(V'_m)$, $m \geq m_0$, that is of the form described, but that satisfies the following. G'_m, $m \geq m_0$, are even-orthogonal groups if and only if G_n, $n \geq n_0$, are symplectic groups. Let χ_G be the character associated to the tower G_n, $n \geq n_0$ (see Section 1). It is trivial if the tower consists of symplectic groups and it is usual quadratic character of V_{n_0} if the tower consists of even-orthogonal groups. This is a convention that we follow in our papers [M3] and [M5]. It helps to avoid case by case analysis of [M4].

The pair (see Definition 1.1) (G_n, G'_m) is a dual pair in the symplectic group $G(V_n \otimes V'_m)$, [MVW, Ku1]. We write

$$\omega_{n,m} = \omega^{\psi}_{n,m},$$

for the smooth oscillator representation associated to that pair and a fixed nontrivial additive character ψ of F.

For $\sigma \in \operatorname{Irr} G_n$, we write $\Theta(\sigma, m)$, for a smooth representation of G'_m, defined as a maximal σ-isotypic quotient of $\omega_{n,m}$ ([MVW])

$$\sigma \otimes \Theta(\sigma, m) \simeq \omega_{n,m} / \bigcap_f \ker(f), \qquad f \in \operatorname{Hom}_{G_n}(\omega_{n,m}|_{G_n}, \ \sigma).$$

We call $\Theta(\sigma, m)$ **the full lift** of σ. It is a smooth representation of G'_m. More precisely, it is a zero or an admissible representation of finite length by Théorème principal in ([MVW], page 69) (see Corollary 3.1 for a different proof). Let us write $m(\sigma)$ for the smallest $m \geq m_0$ such that $\Theta(\sigma, m) \neq 0$.

The first main result of the present paper is the following (Theorem 6.1)

Theorem: *Assume that $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$ is a representation in discrete series. Let*

$$m_{temp}(\sigma) = \begin{cases} m(\sigma); & m(\sigma) > n + \eta_{G'} \\ n + \eta_{G'}; & m(\sigma) \leq n + \eta_{G'}. \end{cases}$$

Then we have the following:

(i) *If m satisfies $m(\sigma) \le m \le m_{temp}(\sigma)$, then all irreducible subquotients of $\Theta(\sigma, m)$ are tempered representations. More precisely, they are all in discrete series if one of the following holds:*
  (1) $m < n + \eta_{G'}$
  (2) $m = m(\sigma) = n + \eta_{G'}$
  (3) $m = m(\sigma) > n + \eta_{G'}$ *and σ does not satisfy that it is a subrepresentation of a induced representation*

$$\chi_{G'} |\det|^{1/2} Steinberg_{GL(2(m-n-\eta_{G'}),\ F)} \rtimes \sigma'',$$

for some representation $\sigma'' \in \operatorname{Irr} G_{n''}$.

(ii) *If $m(\sigma) < n + \eta_{G'}$, then all irreducible subquotients τ of $\Theta(\sigma, n + \eta_{G'})$ are of the form*

$$\tau \hookrightarrow \chi_G \rtimes \tau_1,$$

where τ_1 is an irreducible subquotient of $\Theta(\sigma, n + \eta_{G'} - 1)$ in discrete series.

(iii) *If m satisfies $m > m_{temp}(\sigma)$, then any irreducible quotient $\sigma(m)$ of $\Theta(\sigma, m)$ is a unique irreducible subrepresentation of*

$$|\ |^{n-m+\eta_{G'}} \chi_G \times |\ |^{n-m+\eta_{G'}+1} \chi_G \times \cdots \times |\ |^{n-m_{temp}(\sigma)-\eta_G} \chi_G \rtimes \sigma(m_{temp}(\sigma)),$$

for some irreducible quotient $\sigma(m_{temp}(\sigma))$ of $\Theta(\sigma, m_{temp}(\sigma))$. All other irreducible subquotients of $\Theta(\sigma, m)$ are either tempered or of the form

$$|\ |^{n-m+\eta_{G'}} \chi_G \times |\ |^{n-m+\eta_{G'}+1} \chi_G \times \cdots \times |\ |^{n-m_1-\eta_G} \chi_G \rtimes \sigma(m_1),$$

for some tempered irreducible subquotient $\sigma(m_1)$ of $\Theta(\sigma, m_1)$, where $m > m_1 \ge m_{temp}(\sigma)$.

In fact, a more precise information is available for the shape of possible tempered irreducible subquotients. (See Theorem 4.2).

Now, assume that the residue characteristic of F is different than two. Then the Howe conjecture holds (see [W1]). More precisely, let $\sigma \in \operatorname{Irr} G_n$. Then $\Theta(\sigma, m)$ is zero or it has the unique maximal proper subrepresentation; the corresponding irreducible quotient we denote by $\sigma(m)$.

The following corollary related to the previous theorem (see Corollary 6.1) generalizes (Théorème principal 1, [MVW]) from the case of σ is a supercuspidal representation to the case of general discrete series.

Corollary: *Assume that the residue characteristic of F is different from 2. Let $\sigma \in \operatorname{Irr} G_n$ ($n \ge n_0$) be a representation in discrete series. Then* *there is a unique integer* $m_{temp}(\sigma) \geq n + \eta_{G'}$ *such that* $\sigma(m)$ *is tempered for* $m(\sigma) \leq m \leq m_{temp}(\sigma)$*. Moreover,* $m > m_{temp}(\sigma)$ *we have that* $\sigma(m)$ *is a unique irreducible (Langlands) subrepresentation of*

$$|\ |^{n-m+\eta_{G'}}\chi_G \times |\ |^{n-m+\eta_{G'}+1}\chi_G \times \cdots \times |\ |^{n-m_{temp}(\sigma)-\eta_G}\chi_G \rtimes \sigma(m_{temp}(\sigma)).$$

The next theorem (see Theorem 6.2) describes the structure of the full lifts in important cases. Also, it settles a part of the conjecture made in the introduction of [M4].

Theorem: *Assume that the residue characteristic of* F *is different from* 2*. Let* $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$ *be a representation in discrete series. Then* $\Theta(\sigma, m)$ *is irreducible for* $m(\sigma) \leq m \leq m_{temp}(\sigma)$*. In particular, if* $m \leq n + \eta_{G'}$*, then* $\Theta(\sigma, m)$ *is irreducible or zero.*

This clearly generalizes the corresponding result for supercuspidal representations (see Théorème principal in [MVW, page 69]. In view of Kudla's theory of see–saw pairs it should have important applications to the restriction problems of Gross and Prasad.

The proofs of the main results of the present paper are based on the computation of isotypic components of irreducible representations in Jacquet modules of $\omega_{n,m}$ with respect to maximal parabolic subgroups in G_n using Kudla's filtration of Jacquet modules of $\omega_{n,m}$ [Ku, Ku1]. These computations are done in Section 3 of the present paper (see Theorem 3.1 and Corollary 3.1). They generalize [M4, Proposition 2.1] and again they are based on the existence of a right-adjoint functor to the functor of (normalized) parabolic induction due to J. Bernstein [Be, Be1, Bu].

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1. Preliminaries

Let F be a nonarchimedean field of characteristic different than two. Let $|\ |$ be the (normalized as usual) absolute value of F. Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively.

Let G be an l-group [BZ]. Then, by a representation of G we mean a pair (π, V), where V is a complex vector space and π is a homomorphism $G \to GL(V)$. We write V_∞ for the subspace of V consisting of all vectors in V having open stabilizer in G. Since G is an l-group, V_∞ is $\pi(G)$-invariant; the resulting representation we denote by (π_∞, V_∞). The representation (π, V) is smooth if $V = V_\infty$. We write $\mathcal{A}(G))$ for the category of all smooth complex representations of G. If (π, V) is a smooth representation, then we denote by $(\widetilde{\pi}, \widetilde{V})$ its smooth contragredient representation.

Let $P = MN$ be a closed subgroup of G, given as a semi-direct product of closed subgroups M and N, M normalizes N. Assume that N is a union of its open compact subgroups and G/P is compact. Then we have normalized induction and localization functors $\mathrm{Ind}_P^G : \mathcal{A}(M) \to \mathcal{A}(G)$ and $R_P : \mathcal{A}(G) \to \mathcal{A}(M)$. They are related by the Frobenius reciprocity:

$$\begin{cases} \mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\pi')) \simeq \mathrm{Hom}_M(R_P(\pi), \pi') \\ \mathrm{Hom}_G(\mathrm{Ind}_P^G(\pi'), \pi) \simeq \mathrm{Hom}_M(\pi', \widetilde{R_P(\widetilde{\pi})}) \quad (\pi \text{ is an admissible representation}) \end{cases}$$

Assume that G and G' are l-groups. Let V be a smooth representation of $G \times G'$. If $\rho \in \mathrm{Irr}\, G$ is an admissible representation, then we write $\Theta(\rho, V) \in \mathcal{A}(G')$ for the ρ-isotypic quotient of V (cf. [MVW], Chapitre II, Lemme III.4). More precisely, set $V' = \bigcap_f \ker(f)$, $f \in \mathrm{Hom}_G(V, \rho)$, then

$$V/V' \simeq \rho \otimes \Theta(\rho,\ V).$$

For convenience, let us state the next simple lemma. The proof is left to the reader.

Lemma 1.1: *The (possibly non-smooth) dual representation $\Theta(\rho, V)^*$ of G' is isomorphic to the obvious (not necessarily smooth) representation of G' on $\mathrm{Hom}_G(V, \rho)$. Hence, we have an isomorphism of the corresponding smooth representations $\widetilde{\Theta(\rho, V)} = \Theta(\rho, V)^*_\infty \simeq \mathrm{Hom}_G(V, \rho)_\infty$.*

The next result of Bernstein will be the cornerstone of our analysis of the Howe corresppondence. (See [Be1].)

THEOREM 1.1: *Assume that an l-group G' is a semidirect product $G \rtimes \mathbb{Z}/2\mathbb{Z}$, where G is a connected reductive F-group. Let $P = MN$ be a parabolic subgroup of G, and let $\overline{P} = M\overline{N}$ be the opposite parabolic subgroup of P. Assume that $\mathbb{Z}/2\mathbb{Z}$ normalizes M, N and $\overline{N}$. Put $M' = M \rtimes \mathbb{Z}/2\mathbb{Z}$, $P' = M'N$, and $\overline{P}' = M'\overline{N}$. If $\pi \in \mathcal{A}(M')$ and $\Pi \in \mathcal{A}(G')$, then we have an isomorphism $\phi \mapsto \phi_0$*

$$\mathrm{Hom}_{G'}(\mathrm{Ind}_{P'}^{G'}(\pi), \Pi) \simeq \mathrm{Hom}_{M'}(\pi, R_{\overline{P}'}(\Pi)),$$

where ϕ_0 is given by the composition of the natural inclusion (through a part of filtration that corresponds to a open orbit $P'\overline{P}'$ in $P' \setminus G'$)

$$\pi \hookrightarrow R_{\overline{P}'}(\mathrm{Ind}_{P'}^{G'}(\pi)),$$

and the natural map $\phi : R_{\overline{P}'}(\mathrm{Ind}_{P'}^{G'}(\pi)) \to R_{\overline{P}'}(\Pi)$.

Proof. If G' is connected, then Bernstein has shown that the map $\phi \mapsto \phi_0$ is an isomorphism. Now, the theorem follows, considering the restriction of all representations in question to G. ∎

The next theorem is well-known (see [Be1, Lemma 26.]).

THEOREM 1.2: *Assume that an l-group G' is a semidirect product $G \rtimes \mathbb{Z}/2\mathbb{Z}$, where G is a connected reductive F-group. Assume that $\rho \in \mathcal{A}(G')$ is an irreducible supercuspidal representation, that is, $\rho|_G$ is an (completely reducible) supercuspidal representation, and $\Pi \in \mathcal{A}(G')$ is an admissible representation of finite length. Then, if ρ appears in the composition series of Π, then there exists G-equivariant epimorphism $\Pi \twoheadrightarrow \rho$.*

Proof. The theorem follows, considering the restriction of all representations in question to G. In fact one can adjust the proof of [Be1, Lemma 26] easily. We leave the simple verification to the reader. ∎

Next, we shall describe the groups we consider. We look at usual towers of even-orthogonal or symplectic groups $G_n = G(V_n)$ that are groups of isometries of F-spaces $(V_n, (\ ,\ ))$, $n \geq n_0$, where the form $(\ ,\ )$ is non-degenerate and it is skew-symmetric if the tower is symplectic and symmetric otherwise.

The tower $(V_n, (\ ,\ ))$, $n \geq n_0$, can be described explicitly as follows. We fix an anisotropic F-space $(V_{n_0}, (\ ,\ ))$ of dimension $2n_0 = 0, 2, 4$. (This defines

n_0.) In the case of orthogonal groups G_n, we let $\chi_G = \chi_{V_{n_0}}$ be the quadratic character of $F^\times$ associated to the quadratic space V_{n_0}. (See [Ku], page 240, (2.5), or [Ku1, Proposition 4.3.].) If V_{n_0} is trivial or four dimensional space, then χ_G is the trivial character. In the case of symplectic groups G_n, we let χ_G be the trivial character.

Next, for any $n \in \mathbb{Z}_{\geq n_0}$, let V_n be the orthogonal direct sum of V_{n_0} with $r := n - n_0$ hyperbolic planes. We see $2n = \dim V_n$. We fix a Witt decomposition

$$V_n = V^{(1)} \oplus V_{n_0} \oplus V^{(2)}, \tag{1.1}$$

where $V^{(i)} = Fv_1^{(i)} \oplus \cdots \oplus Fv_r^{(i)}$, $i = 1, 2$, satisfying $(v_k^{(i)}, v_l^{(i)}) = 0$ and $(v_k^{(1)}, v_l^{(2)}) = \delta_{kl}$.

The decomposition (1.1) gives us the set of a standard parabolic subgroups in G_n. We will describe maximal parabolic subgroups. For j, $1 \leq j \leq r$, let $V_j^{(i,n)} = Fv_{r-j+1}^{(i)} \oplus \cdots \oplus Fv_r^{(i)}$, $i = 1, 2$. Then we have the Witt decomposition

$$V_n = V_j^{(1,n)} \oplus V_{n-j} \oplus V_j^{(2,n)}.$$

Let P_j be the parabolic subgroup of G_n which stabilizes $V_j^{(1,n)}$. There is a Levi decomposition $P_j = M_j N_j$, where $M_j \simeq GL(V_j^{(1,n)}) \times G_{n-j}$. (Beware of the difference between this choice of a Levi factor and that of [Ku, page 233]. There is considered $GL(V_j^{(2,n)})$ instead of $GL(V_j^{(1,n)})$.) Fix the isomorphism $GL(j, F) \simeq GL(V_j^{(1,n)})$ using the above fixed basis of $V_j^{(1,r)}$.

We end the discussion of classical groups by introducing more notation:

Definition 1.1: We fix tower of groups $G'_m = G(V'_m)$, $m \geq m_0$, that is, of the form described above but satisfies the following G'_m, $m \geq m_0$, are even-orthogonal groups if and only if G_n, $n \geq n_0$, are symplectic groups. Through the paper we will write $\chi_{G'} = \chi_{V'_{m_0}}$, $\chi_G = \chi_{V_{n_0}}$ and

$$\eta_G = \begin{cases} 0 \text{ if } G_n \text{ is a symplectic group} \\ 1 \text{ if } G_n \text{ is an even-orthogonal group.} \end{cases}$$

Similarly, we define $\eta_{G'}$.

Now, we turn to the representation theory of classical groups. If $\pi \otimes \sigma$ is a smooth representation of $M_j \simeq GL(V_j^{(1,n)}) \times G_{n-j}$, then we write

$$\pi \rtimes \sigma := \operatorname{Ind}_{P_j}^{G_n}(\pi \otimes \sigma),$$

following Tadić.

The next theorem will strength the Frobenius reciprocity. (See [M3, Lemma 1.1.])

LEMMA 1.2: *Let $\sigma \in \operatorname{Irr} G_n$ and let P_j be a maximal parabolic subgroup of G_n. Let $\pi \otimes \sigma''$ be an irreducible representation of $M_j \simeq GL(j,F) \times G_{n-j}$. Next, assume that $\widetilde{\pi} \otimes \sigma'' \hookrightarrow \widetilde{R_{P_j}(\widetilde{\sigma})}$. (For example, this holds if $\widetilde{\pi} \rtimes \sigma'' \twoheadrightarrow \sigma$.) Then $\sigma \hookrightarrow \pi \rtimes \sigma''$.*

At some point in the paper we will need Tadić's theory of Jacquet modules. We end this section recalling his basic result. Let $R(G_n)$ be the Grothendieck group of admissible representations of finite length of G_n. Put

$$R(G) = \bigoplus_{n \geq n_0} R(G_n).$$

We will write $\geq$ or $\leq$ for the natural order on $R(G)$. In more details, $\pi_1 \leq \pi_2$, $\pi_1, \pi_2 \in R(G)$, if and only if $\pi_2 - \pi_1$ is a linear combination of the irreducible representations with positive coefficients. Similarly, we define

$$R(GL) = \bigoplus_{n \geq 0} R(GL(n,F)).$$

For a standard maximal parabolic subgroup P_j of G_n, $1 \leq j \leq n - n_0$, we can identify $R_{P_j}(\sigma)$ with its semisimplification in $R(GL(j,F)) \otimes R(G_{n-j})$. Thus, we can consider

$$\mu^*(\sigma) = \mathbf{1} \otimes \sigma \; + \sum_{j=1}^{n-n_0} R_{P_j}(\sigma) \in R(GL) \otimes R(G),$$

where $\mathbf{1}$ is the trivial representation of the trivial group $GL(0,F)$. We also extend usual induction (see [Ze]) defining $\mathbf{1} \times \pi := \pi$ and $\pi \times \mathbf{1} := \pi$, for every smooth representation π of some $GL(m_\pi, F)$, and $\mathbf{1} \times \mathbf{1} = \mathbf{1}$. Also, we let $\mathbf{1} \rtimes \sigma_1 := \sigma_1$, for every smooth representation σ_1 of G_{n_1}.

We introduce more notation. Assume that $l_1, l_2 \in \mathbb{R}$, $l_1 + l_2 \in \mathbb{Z}$, and $\rho \in \operatorname{Irr} GL(m_\rho, F)$ (this defines m_ρ) is a supercuspidal representation. Then we let (see [Ze])

- $\delta([|\det|^{-l_1}\rho, |\det|^{l_2}\rho])$ be the unique irreducible subrepresentation of
$$|\det|^{l_2}\rho \times |\det|^{l_2-1}\rho \times \cdots \times |\det|^{-l_1}\rho$$
if $l_1 + l_2 \geq 0$.
- $\delta([|\det|^{-l_1}\rho, |\det|^{l_2}\rho]) = \mathbf{1}$ if $l_1 + l_2 < 0$.

The basic result of Tadić is the following theorem (see for example [MT] and reference therein).

Theorem 1.3: *Let $l_1 + l_2 \geq 0$. Let us decompose $\mu^*(\sigma) = \sum_{\delta',\sigma_1} \delta' \otimes \sigma_1$ into irreducible constituents in $R(G)$. Then*

$$\mu^*\left(\delta([|\det|^{-l_1}\rho, |\det|^{l_2}\rho]) \rtimes \sigma\right) = \sum_{\delta',\sigma_1} \sum_{i=0}^{l_1+l_2+1} \sum_{j=0}^{i} \delta([|\det|^{i-l_2}\widetilde{\rho}, |\det|^{l_1}\widetilde{\rho}]) \times \delta([|\det|^{l_2+1-j}\rho, |\det|^{l_2}\rho]) \times \delta' \otimes \delta([|\det|^{l_2+1-i}\rho, |\det|^{l_2-j}\rho]) \rtimes \sigma_1.$$

2. Review of the Theta Correspondence

In this section we review some results about the theta correspondence and fix the notation.

The pair (see Definition 1.1) (G_n, G'_m) is a dual pair in the symplectic group $G(V_n \otimes V'_m)$ [MVW, Ku1]. We write

$$\omega_{n,m} = \omega^{\psi}_{n,m},$$

for the smooth oscillator representation associated to that pair and a fixed nontrivial additive character ψ of F.

For every $\sigma \in \operatorname{Irr} G_n$, we write $\Theta(\sigma, m)$, for a smooth representation of G'_m, defined as a maximal σ-isotypic quotient of $\omega_{n,m}$ [MVW, Chapitre II, Lemme III.4]

$$\sigma \otimes \Theta(\sigma, m) \simeq \omega_{n,m} / \bigcap_f \ker(f), \quad f \in \operatorname{Hom}_{G_n}(\omega_{n,m}|_{G_n},\ \sigma).$$

The basic result about the Howe correspondence is the following theorem [MVW, Théorème principal and Remarque page 67]:

Theorem 2.1: *Let $\sigma \in \operatorname{Irr} G_n$. Then the following hold:*

(i) *There exists a nonnegative integer m such that $\Theta(\sigma, m) \neq 0$. The smallest m such that $\Theta(\sigma, m) \neq 0$ we denote by $m(\sigma)$. Further, for $m \geq m(\sigma)$, we have $\Theta(\sigma, m) \neq 0$. $m(\sigma)$ is called the first occurrence index of σ in the tower G'_m, $m \geq m_0$.*

(ii) *Assume that σ is a supercuspidal representation. Then $\Theta(\sigma, m(\sigma))$ is a supercuspidal irreducible representation, and, for $m \geq m(\sigma)$, $\Theta(\sigma, m)$ is* *an irreducible subrepresentation of*

$$\chi_G|\ |^{n-m+1-\eta_G} \times \cdots \times \chi_G|\ |^{n-m(\sigma)-\eta_G} \rtimes \Theta(\sigma, m(\sigma)).$$

The Jacquet module $R_{P'_{m-m(\sigma)}}(\Theta(\sigma,m))$ *is isomorphic to*

$$\chi_G|\det|^{n-\frac{m+m(\sigma)-1}{2}-\eta_G} \otimes \Theta(\sigma, m(\sigma)).$$

The next theorem that we need gives Kudla's filtration of Jacquet modules of the oscillator representation [Ku].

THEOREM 2.2: *Let* P_k $(1 \le k \le n-n_0)$ *be the standard maximal parabolic subgroup of* G_n. *Then* $R_{P_k}(\omega_{n,m})$ *has a filtration of smooth* $GL(k,F)\times G_{n-k}\times G'_m$*-representations:*

$$0 = J_{k+1} \subset J_k \subset \cdots \subset J_0 = R_{P_k}(\omega_{n,m}),$$

where $J_j/J_{j+1} \simeq J_{kj}$, $0 \le j \le k$, *and*

$$\begin{cases} J_{k0} = \chi_{G'}|\det|^{m-n+\frac{k-1}{2}+\eta_G} \otimes \omega_{n-k,m} \quad \textit{(quotient)} \\ J_{kj} = \mathrm{Ind}_{P_{kj}\ \times\ G_{n-k}\times\ P'_j}^{GL(k,F)\times G_{n-k}\times G'_m}(\Psi_{kj}\otimes\Sigma_j\otimes\omega_{n-k,m-j}), \quad 0<j<k,\ j\le m-m_0 \\ J_{kk} = \mathrm{Ind}_{GL(k,F)\times G_{n-k}\times P'_k}^{GL(k,F)\times G_{n-k}\times G'_m}(\Sigma_k\otimes\omega_{n-k,m-k}), \quad k\le m-m_0 \\ J_{kj} = 0,\ 1\le j\le k, \quad j>m-m_0. \end{cases}$$

Here P_{kj} *is the standard parabolic subgroup of* $GL(k,F)$ *which corresponds to the partition* $(k-j,j)$, $\Psi_{kj} = \chi_{G'}|\det|^{m-n+\frac{k-j-1}{2}+\eta_G}$ *is a character of* $GL(k-j,F)$, *and* Σ_j *is the twist of the standard representation of* $GL(j,F)\times GL(j,F)$ *on smooth locally constant compactly supported complex valued functions* $C_c^\infty(GL(j,F))$:

$$\Sigma_j(g_1,g_2)f(h) = |\det g_1|^{(-1)^{\eta_G}(\eta_G\cdot n+\eta_{G'}\cdot m-\eta_G\cdot k)-\frac{j+1}{2}}$$
$$\times |\det g_2|^{(-1)^{\eta_{G'}}(\eta_G\cdot n+\eta_{G'}\cdot m-\eta_G\cdot k)+\frac{j+1}{2}}\chi_G(\det g_2)\chi_{G'}(\det g_2)f(g_1^{-1}hg_2).$$

(Here the first $GL(j,F)$ *(resp., the second) is a part of the Levi factor of* P_{kj} *(resp., Levi factor of* P'_j*).) Finally, the representation* $\Psi_{kj}\otimes\Sigma_j\otimes\omega_{n-k,m-j}$ *of* $GL(k-j,F)\times GL(j,F)\times GL(j,F)\times G_{n-k}\times G'_{m-j}$ *is extended to a representation of* $P_{kj}\ \times\ G_{n-k}\times\ P'_j$ *trivial over the corresponding unipotent radicals.*

In order to simplify formulation of many statements and to write formulas in a uniform way, we let $G_{n_0} = G_{n_0-j}$ and $G'_{m_0} = G'_{m_0-j}$ for $j \ge 0$. Next, for $n \ge n_0$ and $m \ge m_0$, we let $P_j = M_j = GL(j,F)\times G_{n-j}$ and $N_j = \{1\}$, for

$j > n - n_0$, $P'_j = M'_j = G'_{m-j}$ and $N'_j = \{1\}$, for $j > m - m_0$. Finally, we let $\omega_{n,m} = 0$ if $n < n_0$ or $m < m_0$.

Let $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$. Then it is clear that $\Theta(\sigma, m) = 0$ if $m < m_0$ since $\Theta(\sigma, m)$ is σ-isotypic component of $\omega_{n,m} = 0$. In particular, if $\Theta(\sigma, m) \neq 0$, then $m \geq m_0$.

Although, P'_j ,$j > m - m_0$, is not a subgroup of G'_m $(m \geq m_0)$ we let

$$\begin{cases} \operatorname{Ind}_{P_{kj} \times G_{n-k} \times P'_j}^{GL(k,F) \times G_{n-k} \times G'_m} (\Psi_{kj} \otimes \Sigma_j \otimes \omega_{n-k,m-j}) = 0 \\ \operatorname{Ind}_{GL(k,F) \times G_{n-k} \times P'_k}^{GL(k,F) \times G_{n-k} \times G'_m} (\Sigma_k \otimes \omega_{n-k,m-k}) = 0. \end{cases}$$

Now, the formula for J_{kj} is the same in all cases $0 < j \leq k$. This was used implicitly in [M3] and it simplifies the exposition.

3. Isotypic Components

This section is the technical heart of the paper. The main results are Theorem 3.1 and two of its corollaries (Corollaries 3.1 and 3.2) below. We suggest that the reader skip this section on the first reading.

Now, we fix the notation used in this section. We let $n \in \mathbb{Z}_{>n_0}$. Let $k \in \mathbb{Z}$, $1 \leq k \leq n - n_0$. So that, G_n has a maximal parabolic subgroup P_k with Levi $M_k \simeq GL(k, F) \times G_{n-k}$. Let $\delta \otimes \sigma_1 \in \operatorname{Irr} M_k$.

We consider the tower G'_m, $m \geq m_0$, given by Definition 1.1, and compute the filtration of $\Theta(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m}))$ using the filtration of $R_{P_k}(\omega_{n,m})$ given by Theorem 2.2. We start with the following

Lemma 3.1: *$\Theta(\delta \otimes \sigma_1, J_{kk}) \neq 0$ if and only if $\Theta(\sigma_1, m - k) \neq 0$. Moreover, if this is so, then*

$$\Theta(\delta \otimes \sigma_1, J_{kk}) \simeq \chi_G \chi_{G'} \widetilde{\delta} \rtimes \Theta(\sigma_1, m - k).$$

Proof. Set $\tau = \Theta(\sigma_1, m - k)$ and $\Pi = \Theta(\delta \otimes \sigma_1, J_{kk})$. We note that

$$\begin{cases} \delta \rtimes \sigma_1 = \operatorname{Ind}_{P_k}^{G_n} (\delta \otimes \sigma_1) \\ \chi_G \chi_{G'} \widetilde{\delta} \rtimes \tau = \operatorname{Ind}_{P'_k}^{G'_m} (\chi_G \chi_{G'} \widetilde{\delta} \otimes \tau), \quad k \leq m - m_0. \end{cases}$$

We write $\overline{P}'_k = M'_k \overline{N}'_k$ for a parabolic subgroup of G'_m opposite to $P'_k = M'_k N'_k$ $(k \leq m - m_0)$.

We begin the proof of the lemma. Assume that $\tau \neq 0$. In particular, $k \leq m - m_0$. (See the end of Section 2.) Then, using the notation introduced in

Theorem 2.2, we have an $GL(k,F)\times G_{n-k}\times GL(k,F)\times G'_{m-k}$-equivariant epimorphism

$$\varphi:\Sigma_k\otimes\omega_{n-k,m-k}\twoheadrightarrow\delta\otimes\sigma_1\otimes\chi_G\chi_{G'}\widetilde{\delta}\otimes\tau.$$

Now, inducing we obtain

$$J_{kk}=\mathrm{Ind}_{GL(k,F)\times G_{n-k}\times P'_k}^{GL(k,F)\times G_{n-k}\times G'_m}(\Sigma_k\otimes\omega_{n-k,m-k})\twoheadrightarrow\delta\otimes\sigma_1\otimes\mathrm{Ind}_{P'_k}^{G'_m}(\chi_G\chi_{G'}\widetilde{\delta}\otimes\tau)$$

$GL(k,F)\times G_{n-k}\times G'_{m-k}$-equivariant epimorphism that we denote by $\mathrm{Ind}(\varphi)$. In particular, we see $\Pi\neq 0$. Also, applying the notation introduced in Theorem 1.1 we easily obtain

$$\mathrm{Ind}(\varphi)_0=\varphi.$$

Clearly, $\mathrm{Ind}(\varphi)$ must factor through the canonical $GL(k,F)\times G_{n-k}\times G'_{m-k}$-equivariant epimorphism

$$\psi:J_{kk}\twoheadrightarrow\delta\otimes\sigma_1\otimes\Pi.$$

More precisely, there exists $GL(k,F)\times G_{n-k}\times G'_{m-k}$-equivariant morphism

$$\varphi_1:\delta\otimes\sigma_1\otimes\Pi\to\delta\otimes\sigma_1\otimes\mathrm{Ind}_{P'_k}^{G'_m}(\chi_G\chi_{G'}\widetilde{\delta}\otimes\tau)$$

such that

$$\mathrm{Ind}(\varphi)=\varphi_1\circ\psi. \tag{3.1}$$

Next, as in Theorem 1.1, we can define $GL(k,F)\times G_{n-k}\times GL(k,F)\times G'_{m-k}$-equivariant morphism

$$\psi_0:\Sigma_k\otimes\omega_{n,m}\to\delta\otimes\sigma_1\otimes R_{\overline{P}'_k}(\Pi).$$

Hence, we can choose φ so that we can factor $\psi_0=\varphi'\circ\varphi$, where φ' is $GL(k,F)\times G_{n-k}\times GL(k,F)\times G'_{m-k}$-equivariant morphism

$$\varphi':\delta\otimes\sigma_1\otimes\chi_G\chi_{G'}\widetilde{\delta}\otimes\tau\to\delta\otimes\sigma_1\otimes R_{\overline{P}'_k}(\Pi).$$

Let

$$\varphi'_1:\delta\otimes\sigma_1\otimes\mathrm{Ind}_{P'_k}^{G'_m}(\chi_G\chi_{G'}\widetilde{\delta}\otimes\tau)\to\delta\otimes\sigma_1\otimes\Pi$$

be $GL(k,F)\times G_{n-k}\times G'_{m-k}$-equivariant morphism from Theorem 1.1, such that $(\varphi'_1)_0=\varphi'$. It is not difficult to see

$$\psi_0=(\varphi'_1\circ\mathrm{Ind}(\varphi))_0.$$

Hence, by Theorem 1.1, we obtain

$$\psi=\varphi'_1\circ\mathrm{Ind}(\varphi). \tag{3.2}$$

Now, combining (3.1) and (3.2) we obtain

$$\begin{cases} \psi = \varphi_1' \circ \varphi_1 \circ \psi \\ \mathrm{Ind}(\varphi) = \varphi_1 \circ \varphi_1' \circ \mathrm{Ind}(\varphi). \end{cases} \tag{3.3}$$

Since ψ and $\mathrm{Ind}(\varphi)$ are epimorphisms, we see that (3.3) implies that φ_1 and φ_1' are mutually inverse $GL(k,F) \times G_{n-k} \times G'_{m-k}$-equivariant isomorphisms of $\delta \otimes \sigma_1 \otimes \mathrm{Ind}_{P'_k}^{G'_m}(\chi_G \chi_{G'} \widetilde{\delta} \otimes \tau)$ and $\delta \otimes \sigma_1 \otimes \Pi$. ∎

Lemma 3.2: *Let $0 < j < k$. Using the notation introduced in Theorem 2.2, we let $\overline{P}_{kj}$ be the standard parabolic subgroup of $GL(k,F)$ opposite to P_{kj} and write $R_{\overline{P}_{kj}}(\delta)(\Psi_{kj})$ for the maximal subspace of $R_{\overline{P}_{kj}}(\delta)$ where $GL(k-j,F)$ acts as a character Ψ_{kj}. If nonzero, $R_{\overline{P}_{kj}}(\delta)(\Psi_{kj})$ is an admissible representation of $GL(j,F)$. We assume that $R_{\overline{P}_{kj}}(\delta)(\Psi_{kj})$ is irreducible if nonzero. (This holds if for example δ is an essentially square-integrable representation [Ze].) Then $\Theta(\delta \otimes \sigma_1, J_{kj}) \neq 0$ if and only if $R_{\overline{P}_{kj}}(\delta)(\Psi_{kj}) \neq 0$ and $\Theta(\sigma_1, m-j) \neq 0$. Moreover, if $\Theta(\delta \otimes \sigma_1, J_{kj}) \neq 0$, then we have*

$$\Theta(\delta \otimes \sigma_1, J_{kj}) \simeq \chi_G \chi_{G'} \widetilde{R_{\overline{P}_{kj}}(\delta)(\Psi_{kj})} \rtimes \Theta(\sigma_1, m-j).$$

Remark 3.1: Let us write $R_{P_{kj}}(\widetilde{\delta})_{\Psi_{kj}^{-1}}$ for the maximal quotient of $R_{P_{kj}}(\widetilde{\delta})$, where $GL(k-j,F)$ acts as a character Ψ_{kj}^{-1}. Then the formula for a contragredient of a Jacquet module (see [Ca]) $\widetilde{R_{\overline{P}_{kj}}(\delta)} \simeq R_{P_{kj}}(\widetilde{\delta})$ implies $\widetilde{R_{\overline{P}_{kj}}(\delta)(\Psi_{kj})} \simeq R_{P_{kj}}(\widetilde{\delta})_{\Psi_{kj}^{-1}}$. Thus, we can write

$$\Theta(\delta \otimes \sigma_1, J_{kj}) \simeq \chi_G \chi_{G'} R_{P_{kj}}(\widetilde{\delta})_{\Psi_{kj}^{-1}} \rtimes \Theta(\sigma_1, m-j).$$

Proof of the lemma. In this proof we use the following elementary observation repeatedly (see the proof of Remark 6.11 (iii) in [M5]):

Assume that G and H are l-groups and U an irreducible admissible representation of G. Then the mapping $\alpha \mapsto \mathrm{id}_U \otimes \alpha$ induces an isomorphism of vector spaces:

$$\mathrm{Hom}_H(V,\ W) \simeq \mathrm{Hom}_{G \times H}(U \otimes V,\ U \otimes W).$$

In other words, every $\beta \in \mathrm{Hom}_{G \times H}(U \otimes V,\ U \otimes W)$ can be written uniquely as $\beta = \mathrm{id}_U \otimes \alpha$ for $\alpha \in \mathrm{Hom}_H(V,\ W)$.

Now, we begin the proof. First, we recall (see Theorem 2.2)

$$J_{kj} = \mathrm{Ind}_{P_{kj} \times G_{n-k} \times P'_j}^{GL(k,F) \times G_{n-k} \times G'_m}(\Psi_{kj} \otimes \Sigma_j \otimes \omega_{n-k,m-j}).$$

Set $\tau = \Theta(\sigma_1, m-j)$ and $\Pi = \Theta\left(\delta \otimes \sigma_1, J_{kj}\right)$. We write ψ for the cannonical $GL(k,F) \times G_{n-k} \times G'_m$-equivariant epimorphism

$$\psi : J_{kj} \twoheadrightarrow \delta \otimes \sigma_1 \otimes \Pi.$$

Applying Theorem 1.1, we obtain a nontrivial $GL(k-j,F) \times GL(j,F) \times G_{n-k} \times GL(j,F) \times G'_{m-j}$-equivariant morphism

$$\psi_0 : \Psi_{kj} \otimes \Sigma_j \otimes \omega_{n-k,m-j} \to R_{\overline{P}_{k,j}}(\delta) \otimes \sigma_1 \otimes R_{\overline{P}_j}(\Pi). \tag{3.4}$$

We see that if $\Theta\left(\delta \otimes \sigma_1, J_{kj}\right) \neq 0$, then $R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj}) \neq 0$ and $\Theta(\sigma_1, m-j) \neq 0$. Also, the equivariant map (3.4) induces $GL(j,F) \times G_{n-k} \times GL(j,F) \times G'_{m-j}$-equivariant morphism

$$\psi_0' : \Sigma_j \otimes \omega_{n-k,m-j} \to R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj}) \otimes \sigma_1 \otimes R_{\overline{P}_j}(\Pi).$$

This map must factor through the canonical $GL(j,F) \times G_{n-k} \times GL(j,F) \times G'_{m-j}$-equivariant epimorphism (here we use our assumption that $R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj})$ is irreducible)

$$\varphi' : \Sigma_j \otimes \omega_{n-k,m-j} \twoheadrightarrow R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj}) \otimes \sigma_1 \otimes \chi_G \chi_{G'} \widetilde{R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj})} \otimes \tau.$$

Hence the equivariant map given by (3.4) factors through the canonical $GL(k-j,F) \times GL(j,F) \times G_{n-k} \times GL(j,F) \times G'_{m-j}$–equivariant epimorphism

$$\begin{aligned} \varphi := id \otimes \varphi' : \Psi_{kj} \otimes \Sigma_j \otimes \omega_{n-k,m-j} & \\ \twoheadrightarrow \Psi_{kj} \otimes R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj}) \otimes \sigma_1 \otimes \chi_G \chi_{G'} & \widetilde{R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj})} \otimes \tau. \end{aligned}$$

Hence, there exists $GL(k-j,F) \times GL(j,F) \times G_{n-k} \times GL(j,F) \times G'_{m-j}$-equivariant morphism

$$\begin{aligned} \psi'' : \Psi_{kj} \otimes R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj}) \otimes \sigma_1 \otimes \chi_G \chi_{G'} \widetilde{R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj})} \otimes \tau & \\ \to R_{\overline{P}_{k,j}}(\delta) \otimes \sigma_1 \otimes R_{\overline{P}_j}(\Pi) & \end{aligned}$$

such that

$$\psi_0 = \psi'' \circ \varphi. \tag{3.5}$$

To write this explicitly, we note that $GL(k-j,F) \times GL(j,F)$-equivariant embedding

$$\kappa_0 : \Psi_{kj} \otimes R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj}) \hookrightarrow R_{\overline{P}_{k,j}}(\delta)$$

induces a $GL(k,F)$-equivariant epimorphism (see Theorem 1.1)

$$\kappa : \mathrm{Ind}_{P_{k,j}}^{GL(k,F)}(\Psi_{kj} \otimes R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj})) \twoheadrightarrow \delta.$$

Obviously, ψ'' factors through

$$\begin{aligned}\kappa_0 \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} : \Psi_{kj} \otimes R_{\overline{P}_{k,j}}(\delta)(\Psi_{kj})) \otimes \sigma_1 \otimes \chi_G\chi_{G'} R_{\overline{P}_{k,j}} \widetilde{(\delta)(\Psi_{kj})} \otimes \tau \\ \to R_{\overline{P}_{k,j}}(\delta)) \otimes \sigma_1 \otimes \chi_G\chi_{G'} R_{\overline{P}_{k,j}} \widetilde{(\delta)(\Psi_{kj})} \otimes \tau.\end{aligned}$$

More precisely, there exists $GL(k-j,F) \times GL(j,F) \times G_{n-k} \times GL(j,F) \times G'_{m-j}$-equivariant morphism

$$\varphi'' : R_{\overline{P}_{k,j}}(\delta) \otimes \sigma_1 \otimes \chi_G\chi_{G'} R_{\overline{P}_{k,j}} \widetilde{(\delta)(\Psi_{kj})} \otimes \tau \to R_{\overline{P}_{k,j}}(\delta) \otimes \sigma_1 \otimes R_{\overline{P}_j}(\Pi)$$

such that

$$\psi'' = \varphi'' \circ (\kappa_0 \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}).$$

Also, we must have

$$\varphi'' = \mathrm{id} \otimes \mathrm{id} \otimes {\varphi_1}'', \tag{3.6}$$

where ${\varphi_1}''$ is a $GL(j,F) \times G'_{m-j}$-equivariant morphism

$${\varphi_1}'' : \chi_G\chi_{G'} R_{\overline{P}_{k,j}} \widetilde{(\delta)(\Psi_{kj})} \otimes \tau \to R_{\overline{P}_j}(\Pi).$$

Thus, (3.5) can be written as follows:

$$\psi_0 = \varphi'' \circ (\kappa_0 \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \varphi. \tag{3.7}$$

Next, as in the proof of the previous lemma, we denote by $\mathrm{Ind}(\varphi)$ the corresponding induced $GL(k,F) \times G_{n-k} \times G'_m$-equivariant epimorphism

$$\begin{aligned}&\mathrm{Ind}(\varphi) : J_{kj} \twoheadrightarrow \\ &\mathrm{Ind}_{P_{k,j}}^{GL(k,F)}(\Psi_{kj} \otimes R_{\overline{P}_{kj}}(\delta)(\Psi_{kj})) \otimes \sigma_1 \otimes \mathrm{Ind}_{P'_j}^{G'_m}(\chi_G\chi_{G'} R_{\overline{P}_{k,j}} \widetilde{(\delta)(\Psi_{kj})} \otimes \tau).\end{aligned}$$

By the definition of ψ, there exists a $GL(k,F) \times G_{n-k} \times G'_m$-equivariant morphism

$$\varphi_1 : \delta \otimes \sigma_1 \otimes \Pi \to \delta \otimes \sigma_1 \otimes \mathrm{Ind}_{P'_j}^{G'_m}(\chi_G\chi_{G'} R_{\overline{P}_{k,j}} \widetilde{(\delta)(\Psi_{kj})} \otimes \tau)$$

such that

$$(\kappa \otimes \mathrm{id} \otimes \mathrm{id}) \circ \mathrm{Ind}(\varphi) = \varphi_1 \circ \psi. \tag{3.8}$$

Next, let φ_2 be the morphism that corresponds to φ'' by Theorem 1.1: $(\varphi_2)_0 = \varphi''$. Now, using (3.6) and (3.7), it is not difficult to see

$$(\varphi_2 \circ (\kappa \otimes \mathrm{id} \otimes \mathrm{id}) \circ \mathrm{Ind}(\varphi))_0 = \varphi'' \circ (\kappa_0 \otimes \mathrm{id} \otimes \mathrm{id}) \circ \varphi = \psi_0.$$

Hence, by Theorem 1.1, we obtain

$$\psi = \varphi_2 \circ (\kappa \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \mathrm{Ind}(\varphi). \tag{3.9}$$

Finally, using (3.8) and (3.9), we argue as in the proof of the previous lemma to complete the proof. ∎

Lemma 3.3: *$\Theta(\delta \otimes \sigma_1, J_{k0}) \neq 0$ if and only if $\delta \simeq \chi_{G'} |\det|^{m-n+\frac{k-1}{2}+\eta_G}$ and $\Theta(\sigma_1, m) \neq 0$. If $\Theta(\delta \otimes \sigma_1, J_{k0}) \neq 0$, then*

$$\Theta(\delta \otimes \sigma_1, J_{k0}) \simeq \Theta(\sigma_1, m).$$

Proof. Obvious. ∎

Finally we come to the main result of this section.

Theorem 3.1: *Let $n \in \mathbb{Z}_{>n_0}$. Let $k \in \mathbb{Z}$, $1 \leq k \leq n - n_0$. So that, G_n has a maximal parabolic subgroup P_k with Levi $M_k \simeq GL(k, F) \times G_{n-k}$. Let $\delta \otimes \sigma_1 \in \mathrm{Irr}\, M_k$. Assume that $\Theta(\sigma_1, m'')$ is zero or admissible representation of finite length for every $m'' \in \mathbb{Z}_{\geq m_0}$. (This is true for σ_1 supercuspidal by Theorem 2.1 (ii). We will prove in Corollary 3.1 below that this is always the case. See also [MVW], Chapter 3.) We use the notation introduced in Lemma 3.2. Then $\Theta\left(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m})\right)$ has the following filtration of G'_m-representations*

$$0 = \Theta_{k+1} \subset \Theta_k \subset \cdots \subset \Theta_0 = \Theta\left(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m})\right),$$

where we have the following epimorphisms of (possibly zero) G'_m-representations:

$$\begin{cases} \chi_G \chi_{G'} \widetilde{\delta} \rtimes \Theta(\sigma_1, m-k) \twoheadrightarrow \Theta_k \\ \chi_G \chi_{G'} R_{P_{k,j}}(\widetilde{\delta})_{\Psi_{kj}^{-1}} \rtimes \Theta(\sigma_1, m-j) \twoheadrightarrow \Theta_j/\Theta_{j+1}, \quad 0 < j < k \\ \Theta(\sigma_1, m) \twoheadrightarrow \Theta_0/\Theta_1. \ \ (\text{If } \delta \not\simeq \chi_{G'} |\det|^{m-n+\frac{k-1}{2}+\eta_G}, \text{ then } \Theta_0 = \Theta_1.) \end{cases}$$

Proof. We use the filtration of $R_{P_k}(\omega_{n,m})$ introduced in Theorem 2.2:

$$0 = J_{k+1} \subset J_k \subset \cdots \subset J_0 = R_{P_k}(\omega_{n,m})$$

considered as $GL(k,F)\times G_{n-k}$-representation. Using the restriction maps we obtain the following filtration:

(3.10)
$$0=H_0\subset H_1\subset\cdots\subset H_k\subset H_{k+1}=\mathrm{Hom}_{GL(k,F)\times G_{n-k}}(R_{P_k}(\omega_{n,m}),\ \delta\otimes\sigma_1),$$

where

$$H_i=\{f\in \mathrm{Hom}_{GL(k,F)\times G_{n-k}}(R_{P_k}(\omega_{n,m}),\ \delta\otimes\sigma_1):\ f|_{J_j}=0\}\quad 0\le j\le k+1.$$

Further, we have

$$H_{j+1}/H_j\hookrightarrow \mathrm{Hom}_{GL(k,F)\times G_{n-k}}(J_j/J_{j+1},\ \delta\otimes\sigma_1)\quad 0\le j\le k. \tag{3.11}$$

As in Lemma 1.1, (3.10) is a filtration of (possibly nonsmooth) G'_m-representations, and (3.11) is an embedding of (possibly nonsmooth) G'_m-representations. Taking the smooth parts and using Lemma 1.1, we obtain the following filtration:

$$\begin{aligned}0&=(H_0)_\infty\subset(H_1)_\infty\subset\cdots\subset(H_k)_\infty\subset(H_{k+1})_\infty\\&=\mathrm{Hom}_{GL(k,F)\times G_{n-k}}(R_{P_k}(\omega_{n,m}),\ \delta\otimes\sigma_1)_\infty\\&=\Theta(\delta\otimes\sigma_1,R_{P_k}(\omega_{n,m}))\tilde{},\end{aligned}\tag{3.12}$$

and embeddings

$$\begin{aligned}&(H_{j+1})_\infty/(H_j)_\infty\hookrightarrow(H_{j+1}/H_j)_\infty\\&\hookrightarrow \mathrm{Hom}_{GL(k,F)\times G_{n-k}}(J_j/J_{j+1},\ \delta\otimes\sigma_1)_\infty=\Theta(\delta\otimes\sigma_1,J_j/J_{j+1})\tilde{}\quad 0\le j\le k.\end{aligned}\tag{3.13}$$

Now, since $J_{kj}=J_j/J_{j+1}$ (see Theorem 2.2) and, by the assumption of Theorem 3.1, $\Theta(\sigma_1,m'')$ is zero or an admissible representation of finite length for every $m''\in\mathbb{Z}_{\ge m_0}$, we see that Lemmas 3.1, 3.2, 3.3, show that $\Theta(\delta\otimes\sigma_1,J_j/J_{j+1})$ is zero or an admissible representation. Hence

$$\Theta(\delta\otimes\sigma_1,J_j/J_{j+1})\simeq\left(\Theta(\delta\otimes\sigma_1,J_j/J_{j+1})\tilde{}\right)\tilde{}.$$

Then, (3.13) shows that $(H_{j+1})_\infty/(H_j)_\infty$ is admissible. Hence the filtration (3.12) shows that all representations $(H_j)_\infty$, $0\le j\le k+1$, are admissible. In particular,

$$\Theta(\delta\otimes\sigma_1,R_{P_k}(\omega_{n,m}))\simeq\left(\Theta(\delta\otimes\sigma_1,R_{P_k}(\omega_{n,m}))\tilde{}\right)\tilde{},$$

and we can consider (abusing the notation)

$$\Theta_j=\{f\in\Theta(\delta\otimes\sigma_1,R_{P_k}(\omega_{n,m})):\ f|_{(H_j)_\infty=0}\}\quad 0\le j\le k+1.$$

In this way we obtain the filtration of G'_m-representations:

$$0 = \Theta_{k+1} \subset \Theta_k \subset \cdots \subset \Theta_0 = \Theta\left(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m})\right).$$

Finally, it is not difficult to see

$$\Theta_j/\Theta_{j+1} \simeq ((H_{j+1})_\infty/(H_j)_\infty)\tilde{},$$

and the theorem follows from Lemmas 3.1, 3.2, 3.3, and (3.13). ∎

In this corollary we give a different proof of (Théorème principal 2a of [MVW]).

COROLLARY 3.1: *Let $\sigma \in \operatorname{Irr} G_n$. Then $\Theta(\sigma, m)$ is zero or an admissible representation of finite length.*

Proof. The proof is given by induction on n. If $n = n_0$, then σ is supercuspidal and the claim follows from Theorem 2.1 (ii). In general, $n > n_0$, we have the two cases. If σ is supercuspidal, then we proceed as before. Otherwise, we choose smallest possible k such that there exists $\delta \in \operatorname{Irr} GL(k, F)$ and $\sigma_1 \in \operatorname{Irr} G_{n-k}$ satisfying

$$\sigma \hookrightarrow \delta \rtimes \sigma_1.$$

Hence, by the Frobenius reciprocity $R_{P_k}(\sigma) \twoheadrightarrow \delta \otimes \sigma_1$. Assume that $\Theta(\sigma, m) \neq 0$ for some $m \geq m_0$. Then there exists a nontrivial equivariant map: $\omega_{n,m} \twoheadrightarrow \sigma \otimes \Theta(\sigma, m)$. Hence, the exactness of Jacquet functor implies $R_{P_k}(\omega_{n,m}) \twoheadrightarrow R_{P_k}(\sigma) \otimes \Theta(\sigma, m)$. In particular, we have $R_{P_k}(\omega_{n,m}) \twoheadrightarrow \delta \otimes \sigma_1 \otimes \Theta(\sigma, m)$. Hence $\Theta\left(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m})\right) \twoheadrightarrow \Theta(\sigma, m)$. Now, since k is minimal possible δ is a supercuspidal representation (in particular, Lemma 3.2 is applicable) and the filtrations of Theorem 3.1 prove the corollary by induction. ∎

In the present paper we use Theorem 3.1 when $\delta \in \operatorname{Irr} GL(k, F)$ is an essentially square-integrable representation. The following description can be found in [Ze]. δ is attached to the segment $\Delta = [\rho, \ |\det|^l \rho]$ ($\rho \in \operatorname{Irr} GL(k', F)$ is a supercuspidal representation; $l \in \mathbb{Z}_{\geq 0}$), $\delta = \delta(\Delta)$, as the unique irreducible subrepresentation of the induced representation

$$|\det|^l \rho \times \cdots \times |\det|\rho \times \rho. \tag{3.14}$$

We note that

$$(l+1) \cdot k' = k.$$

Let $k > 1$. Using the notation introduced in Theorem 3.1, we have the following description of its normalized Jacquet modules ($0 < j < k$), see [Ze]:

$$R_{P_{kj}}(\delta) = \begin{cases} 0; \quad k' \text{ does not divide } j \\ \delta([|\det|^{j'}\rho, |\det|^{l}\rho]) \otimes \delta([\rho,\ |\det|^{j'-1}\rho]); \quad j/k' = j' \in \mathbb{Z}. \end{cases}$$

Also, $\widetilde{\delta}$ is an essentially square integrable representation attached to the segment $[|\det|^{-l}\widetilde{\rho},\ \widetilde{\rho}]$, and we have the following description of its normalized Jacquet modules ($0 < j < k$):

$$R_{P_{kj}}(\widetilde{\delta}) =$$
$$\begin{cases} 0; \quad k' \text{ does not divide } j \\ \delta([|\det|^{-j''+1}\widetilde{\rho},\ \ \widetilde{\rho}]) \otimes \delta([|\det|^{-l}\widetilde{\rho},\ |\det|^{-j''}\widetilde{\rho}]);\ (k-j)/k' = j'' \in \mathbb{Z}. \end{cases}$$

Next, we have $R_{P_{kj}}(\widetilde{\delta})_{\Psi_{kj}^{-1}} \neq 0$ if and only if

$$\Psi_{kj}^{-1} = (\chi_{G'}|\det|^{m-n+\frac{k-j-1}{2}+\eta_G})^{-1} = \delta([|\det|^{-j''+1}\widetilde{\rho},\ \ \widetilde{\rho}]), \tag{3.15}$$

and if this is so, we have

$$R_{P_{kj}}(\widetilde{\delta})_{\Psi_{kj}^{-1}} = \delta([|\det|^{-l}\widetilde{\rho}, |\det|^{-j''}\widetilde{\rho}]). \tag{3.16}$$

We rewrite (3.15) in an equivalent formulation. Clearly, (3.15) implies that $k - j = 1$. Now, $(k-j)/k' = j'' \in \mathbb{Z}$ implies $k' = j'' = 1$. Finally, (3.15) is equivalent to

$$\rho = \chi_{G'}|\ |^{m-n+\eta_G} \in \operatorname{Irr} GL(1,F) \quad \text{and} \quad j = k-1. \tag{3.17}$$

Now, (3.15) implies $R_{P_{k,j}}(\widetilde{\delta})_{\Psi_{kj}^{-1}} = \delta([|\ |^{-m+n-\eta_G-l}\chi_{G'}, |\ |^{-m+n-\eta_G-1}\chi_{G'}])$, Since $l+1 = k$, this can be rewritten as follows:

$$R_{P_{k,j}}(\widetilde{\delta})_{\Psi_{kj}^{-1}} = \delta([|\ |^{-m+n+1-\eta_G-k}\chi_{G'}, |\ |^{-m+n-\eta_G-1}\chi_{G'}]),$$

Thus, we have proved the following corollary to Theorem 3.1:

Corollary 3.2: *Let $n \in \mathbb{Z}_{>n_0}$, $k \in \mathbb{Z}$, $1 \leq k \leq n-n_0$. Let $\delta \in \operatorname{Irr} GL(k,F)$ be an essentially square-representation attached to the segment $\Delta = [\rho,\ |\det|^{l}\rho]$ ($\rho \in \operatorname{Irr} GL(k',F)$ is a supercuspidal representation; $l \in \mathbb{Z}_{\geq 0}$). Let $\sigma_1 \in \operatorname{Irr} G_{n-k}$. Then we have the two cases:*

(1) *Assume $\rho \not\simeq \chi_{G'}|\ |^{m-n+\eta_G}$. Then $\Theta(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m}))$ is a (possibly zero) quotient of $\chi_G\chi_{G'}\widetilde{\delta} \rtimes \Theta(\sigma_1, m-k)$.*

(2) *Assume* $\rho = \chi_{G'}|\ |^{m-n+\eta_G} \in \operatorname{Irr} GL(1,F)$. *Then* $\Theta\left(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m})\right)$ *has the following filtration (of possibly zero) smooth* G'_m*-representations*

$$0 \subset \Theta_0 \subset \Theta\left(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m})\right),$$

where we have the following:

$$\delta([|\ |^{-m+n+1-\eta_G-k}\chi_G, |\ |^{-m+n-\eta_G}\chi_G]) \rtimes \Theta(\sigma_1, m-k) \twoheadrightarrow \Theta_0$$

$$\delta([|\ |^{-m+n+1-\eta_G-k}\chi_G, |\ |^{-m+n-\eta_G-1}\chi_G]) \rtimes \Theta(\sigma_1, m-k+1) \twoheadrightarrow \Theta\left(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m})\right)/\Theta_0.$$

4. Some general results on the structure of the full lifts

We start this section with the following observation:

LEMMA 4.1: *Let* $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$. *Assume that* $\delta \in \operatorname{Irr} GL(m_\delta, F)$. *Fix an embedding* $\delta \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_l$, *where* $\rho_i \in \operatorname{Irr} GL(m_{\rho_i}, F)$ *are supercuspidal representations. If* $\delta \otimes \tau_1$ *is an irreducible subquotient of* $R_{P'_{m_\delta}}(\Theta(\sigma, m))$, *for some irreducible representation* τ_1, *then there exists an irreducible representation* τ_2 *such that*

$$\operatorname{Hom}_{G'_m}\left(\Theta(\sigma, m), \rho_1 \times \rho_2 \times \cdots \times \rho_l \rtimes \tau_2\right) \neq 0.$$

Proof. Since $\Theta(\sigma, m)$ is zero or it is an admissible representation of finite length, this follows from Theorem 1.2 applying the transitivity of Jacquet modules. ∎

LEMMA 4.2: *Let* $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$. *Assume that* $\Theta(\sigma, m) \neq 0$. *Then all irreducible subquotients of* $\Theta(\sigma, m)$ *have the same supercuspidal support (up to association).*

Proof. We prove the lemma by induction on $n \geq n_0$. First, we proceed exactly as in the proof of Corollary 3.1 (and using the notation introduced there). If $n = n_0$, then σ is supercuspidal and the claim follows from Theorem 2.1 (ii). In general, $n > n_0$, we have the two cases. If σ is supercuspidal, then we proceed as before. Otherwise, we choose smallest possible k such that there exists $\delta \in \operatorname{Irr} GL(k, F)$ and $\sigma_1 \in \operatorname{Irr} G_{n-k}$ satisfying

$$\sigma \hookrightarrow \delta \rtimes \sigma_1.$$

As in Corollary 3.1 we see that δ is supercuspidal and $\Theta\left(\delta \otimes \sigma_1, R_{P_k}(\omega_{n,m})\right) \twoheadrightarrow \Theta(\sigma, m)$. We apply Corollary 3.2 to see that we have the two cases. First, if $\delta \not\simeq \chi_{G'}|\ |^{m-n+\eta_G}$, then $\chi_G\chi_{G'}\widetilde{\delta} \rtimes \Theta(\sigma_1, m-k) \twoheadrightarrow \Theta(\sigma, m)$ proving the induction step. Next, let $\delta \simeq \chi_{G'}|\ |^{m-n+\eta_G}$. Then $\Theta(\sigma, m)$ has the filtration

$$0 \subset \Theta_0 \subset \Theta(\sigma, m),$$

where $|\ |^{-m+n-\eta_G}\chi_G \rtimes \Theta(\sigma_1, m-k) \twoheadrightarrow \Theta_0$ and $\Theta(\sigma_1, m-k+1) \twoheadrightarrow \Theta(\sigma, m)/\Theta_0$. If $\Theta_0 = 0$, then the induction step is again immediate. Assume $\Theta_0 \neq 0$. In particular, $\Theta(\sigma_1, m-k) \neq 0$. If $\Theta(\sigma, m)/\Theta_0 = 0$, then the inductive step is again immediate. Assume $\Theta(\sigma, m)/\Theta_0 \neq 0$. Now, by the induction hypothesis all irreducible subquotients of Θ_0 have the same supercuspidal support. Up to association, it is obtained taking one of the supercuspidal supports common, by the induction hypothesis, to all irreducible subquotients of $\Theta(\sigma_1, m-k)$ and joining $|\ |^{-m+n-\eta_G}\chi_G$. On the other hand, by the induction hypothesis, all irreducible subquotients of $\Theta(\sigma_1, m-k+1)$ have the same supercupidal support up to association. Since, by Theorem 2.2, we have a nontrivial equivariant map $\Theta(\sigma_1, m-k+1) \to |\ |^{-m+n-\eta_G}\chi_G \rtimes \Theta(\sigma_1, m-k)$, up to association, it is obtained taking one of the supercuspidal supports of common, by the induction hypothesis, to all irreducible subquotients of $\Theta(\sigma_1, m-k)$ and joining $|\ |^{-m+n-\eta_G}\chi_G$. The same is true for $\Theta(\sigma, m)/\Theta_0$ proving the induction step. ∎

The next lemma is [M3, Lemma 5.1]. Its (simple) proof uses only Kudla's filtration of Jacquet modules (see Theorem 2.2) and square-integrability criterion from [Ca] (extended to orthogonal groups in [MT]).

Lemma 4.3: *Let $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$ be a representation in discrete series. Assume that $m \geq n + \eta_{G'}$. Then we have the following:*

(i) *$\Theta(\sigma, m-1) \neq 0$ if and only if $R_{P'_1}(\Theta(\sigma, m))$ has an irreducible subquotient of the form $\chi_G|\ |^{n-m+\eta_{G'}} \otimes \tau$.*

(ii) *Let $\sigma(m)$ be an irreducible quotient of $\Theta(\sigma, m)$ such that $R_{P'_1}(\sigma(m))$ has an irreducible subquotient of the form $\chi_G|\ |^{n-m+\eta_{G'}} \otimes \tau$. Then there exists an irreducible quotient $\sigma(m-1)$ of $\Theta(\sigma, m-1)$ such that*

$$\sigma(m) \hookrightarrow \chi_G|\ |^{n-m+\eta_{G'}} \rtimes \sigma(m-1).$$

The first main result of this section is the following theorem.

THEOREM 4.1: *Let $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$ be a representation in discrete series. Assume that $\Theta(\sigma, m) \neq 0$. Then if τ is a nontempered irreducible subquotient of $\Theta(\sigma, m)$, then $m > n + \eta_{G'}$ and $\tau \hookrightarrow \chi_G|\ |^{n-m+\eta_{G'}} \rtimes \tau_1$, where τ_1 is an irreducible subquotient of $\Theta(\sigma, m-1)$. In particular, $\Theta(\sigma, m-1) \neq 0$. If $m \leq n + \eta_{G'}$ or $m = m(\sigma) > n + \eta_{G'}$, then all irreducible subquotients of $\Theta(\sigma, m)$ are tempered.*

Proof. Since τ is nontempered, by the Langlands classification, we can find a supercuspidal unitary representation $\rho \in \operatorname{Irr} GL(m_\rho, F)$, $\alpha, \beta \in \mathbb{R}$, $\beta - \alpha \in \mathbb{Z}_{\geq 0}$, and $\alpha + \beta < 0$ such that

$$\tau \hookrightarrow \delta([|\det|^{\alpha}\rho, |\det|^{\beta}\rho]) \rtimes \tau_1, \tag{4.1}$$

for some irreducible representation τ_1. Since

$$\delta([|\det|^{\alpha}\rho, |\det|^{\beta}\rho]) \hookrightarrow |\det|^{\beta}\rho \times |\det|^{\beta-1}\rho \times \cdots \times |\det|^{\alpha}\rho,$$

Lemma 4.1 implies the existence of an irreducible representation τ_2 and a nontrivial G'_m-equivariant map

$$\psi : \Theta(\sigma, m) \to |\det|^{\beta}\rho \times |\det|^{\beta-1}\rho \times \cdots \times |\det|^{\alpha}\rho \rtimes \tau_2.$$

By [Ze], for any sequence $\beta = \gamma_0 > \gamma_1 > \cdots > \gamma_k = \alpha - 1$ $(k \geq 1)$, the representation

$$\begin{aligned}\delta([|\det|^{\gamma_1+1}\rho, |\det|^{\gamma_0}\rho]) \times \delta([|\det|^{\gamma_2+1}\rho, |\det|^{\gamma_1}]) \\ \times \cdots \times \delta([|\det|^{\gamma_k+1}\rho, |\det|^{\gamma_{k-1}}\rho]) \rtimes \tau_2.\end{aligned} \tag{4.2}$$

may be regarded as a subrepresentation of $|\det|^{\beta}\rho \times |\det|^{\beta-1}\rho \times \cdots \times |\det|^{\alpha}\rho \rtimes \tau_2$ which itself is of that form if we let $\gamma_0 = \beta, \gamma_1 = \beta - 1$ etc. Therefore, we make take the smallest $k \geq 1$ such that there exists a sequence $\beta = \gamma_0 > \gamma_1 > \cdots > \gamma_k = \alpha - 1$ such that the image of ψ is contained in the representation given by (4.2). Since k is minimal possible, we can permute representations $\delta([|\det|^{\gamma_{j+1}+1}\rho, |\det|^{\gamma_j}\rho])$ $(0 \leq j \leq k-1)$ in the expression of (4.2) as we want and the image of ψ is still contained in the resulting representation. In particular, let $0 \leq j \leq k-1$, then there is a nonzero G'_m-equivariant map

$$\Theta(\sigma, m) \to \delta([|\det|^{\gamma_{j+1}+1}\rho, |\det|^{\gamma_j}\rho]) \rtimes \tau_3, \tag{4.3}$$

for some irreducible representation τ_3. Let $\delta([|\det|^{\gamma_{j+1}+1}\rho, |\det|^{\gamma_j}\rho]) \in \operatorname{Irr} GL(m_\delta, F)$ (this defines m_δ). Now, the Frobenius reciprocity, applied to

(4.3), implies

$$R_{P'_{m_\delta}}(\Theta(\sigma, m)) \twoheadrightarrow \delta([|\det|^{\gamma_{j+1}+1}\rho, |\det|^{\gamma_j}\rho]) \otimes \tau_3.$$

Hence

$$R_{P'_{m_\delta}}(\omega_{n,m}) \twoheadrightarrow \sigma \otimes \delta([|\det|^{\gamma_{j+1}+1}\rho, |\det|^{\gamma_j}\rho]) \otimes \tau_3.$$

Applying Corollary 3.2, we obtain that one of the following holds:

$$\delta([|\det|^{-\gamma_j}\chi_G\chi_{G'}\widetilde{\rho}, |\det|^{-\gamma_{j+1}-1}\chi_G\chi_{G'}\widetilde{\rho}]) \rtimes \Theta(m - m_\delta, \tau_3) \twoheadrightarrow \sigma \tag{4.4}$$

or

$$\delta([|\det|^{-\gamma_j}\chi_G\chi_{G'}\widetilde{\rho}, |\det|^{-\gamma_{j+1}-2}\chi_G\chi_{G'}\widetilde{\rho}]) \rtimes \Theta(m - m_\delta + 1, \tau_3) \twoheadrightarrow \sigma, \tag{4.5}$$

where, in the last case, we must have

$$|\det|^{\gamma_{j+1}+1}\rho \simeq |\ |^{n-m+\eta_{G'}}\chi_G. \tag{4.6}$$

Let $j = k-1$. We show that (4.4) is not possible. In more detail, if (4.4) holds, then there exists an irreducible subquotient σ'' of $\Theta(m - m_\delta, \tau_3)$ such that

$$\delta([|\det|^{-\gamma_j}\chi_G\chi_{G'}\widetilde{\rho}, |\det|^{-\gamma_{j+1}-1}\chi_G\chi_{G'}\widetilde{\rho}]) \rtimes \sigma'' \twoheadrightarrow \sigma.$$

Hence, Lemma 1.2 implies

$$\sigma \hookrightarrow \delta([|\det|^{\gamma_{j+1}+1}\chi_G\chi_{G'}\widetilde{\rho}, |\det|^{\gamma_j}\chi_G\chi_{G'}\widetilde{\rho}]) \rtimes \sigma''.$$

Since by definition $\alpha = \gamma_k + 1$, we have

$$\gamma_{k-1} + \gamma_k + 1 \leq \beta + \alpha < 0. \tag{4.7}$$

This contradicts the square-integrability criterion for σ. Thus, (4.4) is not possible for $j = k-1$. Similarly, we show $\gamma_j = \gamma_{j+1} + 1$ in (4.5) for $j = k-1$. In more detail, if this does not hold, then the segment appearing in (4.5) is nontrivial and arguing as before we would obtain

$$\gamma_{k-1} + \gamma_k + 2 = \gamma_{k-1} + \alpha + 1 \leq \beta + \alpha + 1 \leq 0. \tag{4.8}$$

Again, this contradicts the square-integrability criterion for σ. In particular, (4.5) and (4.6) hold with

$$\gamma_{k-1} = \gamma_k + 1 = \alpha = n - m + \eta_{G'}. \tag{4.9}$$

We show $k = 1$. If not, $k \geq 2$, and we may consider $j = k-2$ in (4.3). We show that (4.4) is not possible. For if it does hold, then as before (see (4.7))

$$\gamma_{k-2} + \gamma_{k-1} + 1 = \gamma_{k-2} + (\alpha + 1) \leq \beta + \alpha + 1 \leq 0. \tag{4.10}$$

This contradicts the square-integrability criterion for σ. Thus, (4.5) and (4.6) hold for $j = k-2$. In particular, (4.6) implies $\gamma_{k-1}+1 = n-m+\eta_{G'} = \gamma_k+1$. This contradicts the assumption $\gamma_{k-1} > \gamma_k$. This completes the proof of $k=1$.

Now, (4.9) implies $\beta = \gamma_0 = \gamma_1 + 1 = \alpha$. Next, by our assumption, $\alpha+\beta<0$, we have $\alpha < 0$. Hence (4.9) implies $m = n+\eta_{G'}-\alpha > n+\eta_{G'}$.

Summarizing, we have shown that if τ is not tempered, then we have

$$\tau \hookrightarrow |\ |^{n-m+\eta_{G'}}\chi_G \rtimes \tau_1 \quad \text{and} \quad m > n+\eta_{G'}. \tag{4.11}$$

Now, the Frobenius reciprocity shows $R_{P_1'}(\tau) \twoheadrightarrow |\ |^{n-m+\eta_{G'}}\chi_G \otimes \tau_1$. Since τ is a subquotient of $\Theta(\sigma,m)$, we see $|\ |^{n-m+\eta_{G'}}\chi_G\otimes\tau_1$ is a subquotient of $R_{P_1'}(\Theta(\sigma,m))$. Since $\Theta(\sigma,m)$ is admissible (see Corollary 3.1), $R_{P_1'}(\Theta(\sigma,m))$ is also admissible. Therefore, we can decompose:

$$R_{P_1'}(\Theta(\sigma,m)) = \bigoplus_{\mu} R_{P_1'}(\Theta(\sigma,m))_\mu,$$

where the sum runs over a finite set of characters of $GL(1,F)$ and where $R_{P_1'}(\Theta(\sigma,m))_\mu$ is a maximal subrepresentation of $R_{P_1'}(\Theta(\sigma,m))$ having all irreducible subquotients of the form $\mu\otimes\tau_\mu$ (for various τ_μ). Clearly, $|\ |^{n-m+\eta_{G'}}\chi_G\otimes\tau_1$ is a subquotient of $R_{P_1'}(\Theta(\sigma,m))_{|\ |^{n-m+\eta_{G'}}}$. Since $\sigma\otimes\Theta(\sigma,m)$ is a quotient of $\omega_{n,m}$, we see that $\sigma\otimes R_{P_1'}(\Theta(\sigma,m))$ is a quotient of $R_{P_1'}(\omega_{n,m})$. In particular, $\sigma\otimes R_{P_1'}(\Theta(\sigma,m))_{|\ |^{n-m+\eta_{G'}}}$ is a quotient of $R_{P_1'}(\omega_{n,m})$. Using the filtration of $R_{P_1'}(\omega_{n,m})$ (see Theorem 2.2):

$$\begin{cases} J_{10} = \chi_G|\ |^{-m+n+\eta_{G'}}\otimes\omega_{n,m-1} & \text{(quotient)}\\ J_{11} = \operatorname{Ind}_{P_1\times GL(1,F)\times G'_{m-1}}^{G_n\times GL(1,F)\times G'_{m-1}}(\Sigma_1\otimes\omega_{n-1,m-1}) & \text{(subrepresentation)}\end{cases}$$

and arguing as in [M3, Lemma 3.1] we see that $R_{P_1'}(\Theta(\sigma,m))_{|\ |^{n-m+\eta_{G'}}}$ must be a quotient of J_{10}. Hence τ_1 is a subquotient of $\Theta(\sigma,m-1)$. ∎

The second main result of this section tells us about the structure of possible tempered irreducible subquotients of a full lift of a representation in discrete series.

Theorem 4.2: *Let $\sigma\in\operatorname{Irr}G_n$ ($n\geq n_0$) be a representation in discrete series. Assume that $\Theta(\sigma,m)\neq 0$. Then we have the following:*

(i) *If $m < n+\eta_{G'}$ or $m = m(\sigma) = n+\eta_{G'}$, then all irreducible subquotients of $\Theta(\sigma,m)$ are representations in discrete series.*

(ii) *Assume that* $m = n + \eta_{G'} > m(\sigma)$. *Then all irreducible subquotients of* $\Theta(\sigma, m)$ *are tempered representations. More precisely, any tempered, but not in discrete series, irreducible subquotient* τ *of* $\Theta(\sigma, m)$ *is of the form*

$$\tau \hookrightarrow \chi_G \rtimes \tau_1,$$

where τ_1 *is an irreducible subquotient (in discrete series) of* $\Theta(\sigma, n+\eta_{G'} - 1)$.

(iii) *Assume* $m > n+\eta_{G'}$. *Let* $l = m-n-\eta_{G'}$. *Let* $n'' = n-2l$ *and* $m'' = m-2l$. *Then any tempered, but not in discrete series, irreducible subquotient* τ *of* $\Theta(\sigma, m)$ *is of the form*

$$\tau \hookrightarrow \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \rtimes \tau_1,$$

for some representation τ_1 *in discrete series. It exists only if* $n - n_0 \geq 2l$, $m - m_0 \geq 2l + 1$ *and the following holds:*

$$\sigma \hookrightarrow \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]) \rtimes \sigma'',$$

for some (in fact, for unique by Theorem 5.1 (ii)) representation in discrete series $\sigma'' \in \operatorname{Irr} G_{n''}$, *such that* τ_1 *is an irreducible subquotient of* $\Theta(\sigma'', m''-1)$. *(In particular,* $\Theta(\sigma'', m''-1) \neq 0$.*)*

Proof. We will modify the proof of the previous theorem. Let τ be a tempered, but not in discrete series, irreducible subquotient of $\Theta(\sigma, m)$. We assume that τ is given as the one at the beginning of that proof but we require that τ_1 is also tempered and $\alpha + \beta = 0$ instead of $\alpha + \beta < 0$ (see (4.1)). We look at the equivariant map ψ and define k exactly as in the proof of Theorem 4.1. Again, we look at the G'_m-equivariant map (4.3) and the conclusions (4.4) and (4.5) follow. Note that (4.4) cannot hold for $j = k - 1$ using the argument similar to the one given in the proof of the previous theorem. (In fact, the line (4.7) must be replaced with $\gamma_{k-1} + \gamma_k + 1 \leq \beta + \alpha = 0$. This contradicts the square-integrability criterion for σ.) Thus, (4.5) and (4.6) hold. Consequently, (4.9) holds and $\rho = \chi_G$. There are two cases: $k = 1$ and $k > 1$. If $k = 1$. Then (4.9) implies

$$\beta = \gamma_0 = \gamma_1 + 1 = \alpha = -\beta.$$

Hence $\alpha = \beta = 0$. Then, by (4.9), $m = n + \eta_{G'}$. Now, (4.11) should be replaced with

$$\tau \hookrightarrow |\ |^{n-m+\eta_{G'}}\chi_G \rtimes \tau_1 \quad \text{and} \quad m = n + \eta_{G'}, \tag{4.12}$$

and the discussion after (4.11) in the proof of the previous theorem applies proving (i) and (ii). (Note that the case $k > 1$ does not count here. In more detail, then $\beta = \gamma_0 > \gamma_{k-1} = \alpha = -\beta$ (see (4.9)). Hence $n - m + \eta_{G'} = \alpha < 0$. Thus $m > n+\eta_{G'}$. We apply Theorem 4.1 to see that there are no nontempered subquotients.)

Assume $k > 1$. We use the proof of the previous theorem once more. We apply (4.3) with $j = k-2$. Again, as in the previous proof we show that (4.5) cannot hold. Thus (4.4) hold. We show $k = 2$. If not, then (4.4) implies the following modification of (4.10)

$$\gamma_{k-2} + \gamma_{k-1} + 1 = \gamma_{k-2} + (\alpha+1) < \beta + \alpha + 1 = 1.$$

This contradicts the square–integrability criterion for σ. Thus $k = 2$. We summarize:

$$\beta = \gamma_0 > \gamma_1 = \alpha > \gamma_2 = \alpha - 1 \quad \text{and} \quad l := m - n - \eta_{G'} = -\alpha = \beta > 0.$$

Now, (4.5) (for $j = 0$) implies

$$\sigma \hookrightarrow \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]) \rtimes \sigma'', \tag{4.13}$$

for some irreducible representation σ''. Theorem 5.1 (ii), proved in the next section, shows that σ'' is in discrete series. So far, we have proved all claims in (iii) except the following two claims:

$$\tau_1 \text{ is a representation in discrete series} \tag{4.14}$$

and

$$\tau_1 \text{ is an irreducible subquotient of } \Theta(\sigma'', m''-1). \tag{4.15}$$

We proved them now. First, the Frobenius reciprocity implies

$$R_{P_1}(\sigma) \twoheadrightarrow \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]) \otimes \sigma''.$$

Note that $-l+1 = -m+n+\eta_{G'}+1 \neq m-n+\eta_G$. Hence, Corollary 3.2 implies the following:

$$\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \rtimes \Theta(\sigma'', m'') \twoheadrightarrow \Theta(\sigma, m).$$

Since τ is a subquotient of $\Theta(\sigma, m)$, it is also a subquotient of

$$\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \rtimes \Theta(\sigma'', m'').$$

Hence (see Theorem 1.3 for the notation)

$$\mu^*\left(\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \rtimes \Theta(\sigma'', m'')\right) \geq \mu^*(\tau) \geq \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \otimes \tau_1.$$

(The last inequality follows from $\tau \hookrightarrow \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \rtimes \tau_1$ using Frobenius reciprocity.) Hence

$$\mu^*\left(\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \rtimes \Theta(\sigma'', m'')\right) \geq \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \otimes \tau_1. \tag{4.16}$$

The next lemma proves (4.15).

Lemma 4.4: *If* (4.16) *holds, then* τ_1 *is an irreducible subquotient of* $\Theta(\sigma'', m''-1)$.

Proof. The inequality (4.16) can be analyzed using Theorem 1.3. In particular, we see that there exist an irreducible constituent $\delta' \otimes \tau'$ of $\mu^*(\Theta(\sigma'', m''))$, and indices i, j, $0 \leq j \leq i \leq 2l$, such that

$$\begin{cases} \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \leq \delta([|\ |^{i-l+1}\chi_G, |\ |^{l}\chi_G]) \times \delta([|\ |^{l-j}\chi_G, |\ |^{l-1}\chi_G]) \times \delta' \\ \tau_1 \leq \delta([|\ |^{l-i}\chi_G, |\ |^{l-1-j}\chi_G]) \rtimes \tau' \end{cases} \tag{4.17}$$

Now, we have several cases.

- $i \leq 2l-2$. Then $-l < i-l+1 \leq l-1$. Hence the first formula in (4.17) implies $l-j=l$. Hence $j=0$. Now, the classification of nondegenerate representations (see [Ze]) implies

$$\delta' \simeq \delta([|\ |^{-l}\chi_G, |\ |^{i-l}\chi_G]).$$

Since $\delta' \otimes \tau'$ is an irreducible constituent of $\mu^*(\Theta(\sigma'', m''))$, we obtain a nonzero equivariant map (see Lemma 4.1):

$$\Theta(\sigma'', m'') \to |\ |^{i-l}\chi_G \times |\ |^{i-l-1}\chi_G \times \cdots \times |\ |^{-l}\chi_G \rtimes \tau_1',$$

for some irreducible representation τ_1'. Arguing as in the proof of Theorem 4.1, we conclude that $i=0$. (We sketch the argument. If not, then we call the displayed map ψ and look at the analogue of (4.2). Since $(i-l)+(-l) = i-2l < 0$, we conclude $k=1$ and $i-l=\gamma_0=\gamma_1+1=-l$.)

Now, the second inequality in (4.17) implies $\tau_1 \simeq \tau'$. Furthermore, since $m''-n''-\eta_{G'} = l > 0$, we see $|\ |^{m''-n''-\eta_{G'}}\chi_G \otimes \tau_1$ is an irreducible subquotient of $R_{P_1'}(\Theta(\sigma'', m''))$. Hence, as in the discussion after (4.11) in the proof of previous theorem, we obtain that τ_1 is an irreducible subquotient of $\Theta(\sigma'', m''-1)$. In particular, $\Theta(\sigma'', m''-1) \neq 0$.

- $i = 2l-1$. Then $i-l+1 = l$. Also, $l-j \geq l-i = -l+1$. Now, $\delta' \simeq \delta([|\ |^{-l}\chi_G, |\ |^{l-j-1}\chi_G])$ is not trivial. We argue as in the previous case to conclude $j = 2l - 1$. The same conclusion holds. That is, τ_1 is an irreducible subquotient of $\Theta(\sigma'', m'' - 1)$. In particular, $\Theta(\sigma'', m'' - 1) \neq 0$.

- $i = 2l$. Then $l - j \geq l - i = -l$. Hence

$$\delta' \simeq \begin{cases} \delta([|\ |^{-l}\chi_G, |\ |^{l-j-1}\chi_G]) \times |\ |^l\chi_G; & j > 0 \\ \delta([|\ |^{-l}\chi_G, |\ |^l\chi_G]); & j = 0. \end{cases}$$

Using [Ze], we obtain $\delta' \hookrightarrow |\ |^l\chi_G \times \delta''$ (for some irreducible representation δ''). Since $\mu^*(R_{P'_1}(\Theta(\sigma'', m''))) \geq \delta' \otimes \tau'$, we obtain

$$R_{P'_1}(\Theta(\sigma'', m'')) \twoheadrightarrow |\ |^l\chi_G \rtimes \tau_2,$$

for some irreducible representation τ_2. Now, Corollary 3.2 and Lemma 1.2 imply that $\sigma'' \hookrightarrow |\ |^l\chi_{G'} \rtimes \sigma_2$, for some irreducible representation σ_2. Applying (4.12), this contradicts Theorem 5.1 (ii). ∎

It remains to prove (4.14). We remark that $l = m'' - n'' - \eta_{G'} = m - n - \eta_{G'} > 0$. Let $l'' = l - 1 \geq 0$. Since $\tau_1 \in \operatorname{Irr} G'_{m''-1}$ is tempered by the assumption (stated at the beginning of this proof), if we assume that it is not in discrete series, then by the already established parts of Theorem 4.2 we have

$$\tau_1 \hookrightarrow \delta([|\ |^{-l''}\chi_G, |\ |^{l''}\chi_G]) \rtimes \tau_2,$$

for some tempered irreducible representation τ_2. Combining this with $\tau \hookrightarrow \delta([|\ |^{-l}\chi_G, |\ |^l\chi_G]) \rtimes \tau_1$, we obtain

$$\tau \hookrightarrow \delta([|\ |^{-l''}\chi_G, |\ |^{l''}\chi_G]) \rtimes \tau_3,$$

for some tempered irreducible representation τ_3. Since $l = m - n - \eta_{G'} \neq l''$, Corollary 3.2 implies

$$\sigma \hookrightarrow \delta([|\ |^{-l''}\chi_{G'}, |\ |^{l''}\chi_{G'}]) \rtimes \sigma_1,$$

for some irreducible representation σ_1. This contradicts the square-integrability criterion for σ. ∎

5. Some results on discrete series

In this section we reprove some technical results on discrete series ([Mœ], [MT]). We give the proofs that does not use the hypotesis made there. We suggest that the reader skip this section on the first reading and proceed directly to the next section.

THEOREM 5.1: *Assume that $\sigma \in \operatorname{Irr} G_n$ ($n \geq n_0$) is a representation in discrete series. Then we have the following:*

(i) *If an irreducible tempered representation share a supercuspidal support with σ, then it is in discrete series.*

(ii) *Assume that*

$$\sigma \hookrightarrow \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]) \rtimes \sigma'',$$

for some irreducible representation σ'' and $l \in \mathbb{Z}_{>0}$. Then σ'' is a representation in discrete series and there is no irreducible representation σ_1 such that $\sigma'' \hookrightarrow |\ |^{l}\chi_{G'} \rtimes \sigma_1$. Moreover, if $\mu^*(\sigma) \geq \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]) \otimes {\sigma_0}''$, for some irreducible representation ${\sigma_0}''$, then $\sigma'' \simeq {\sigma_0}''$.

Proof. We prove (i). The arguments used here are similar to the one used in [M1] and [M2]. Assume that a tempered representation τ share a supercuspidal support with σ. If τ is not in discrete series, then we can find a supercuspidal unitary representation $\rho \in \operatorname{Irr} GL(m_\rho, F)$, $2\alpha \in \mathbb{Z}_{\geq 0}$ such that

$$\tau \hookrightarrow \delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho]) \rtimes \tau_1, \tag{5.1}$$

for some tempered irreducible representation τ_1. Now, since τ and σ have the same supercuspidal support, a basic result of Tadić [T] (on a supercuspidal support of a representation in discrete series) implies

$$\widetilde{\rho} \simeq \rho. \tag{5.2}$$

Now, using the multiplicative properties of Plancherel factors (see [W2]) it is not hard to see

$$\mu(s, \delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho]) \otimes \tau) = \mu(s, \delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho]) \otimes \sigma). \tag{5.3}$$

Also, (5.1) and (5.2) imply

$$\begin{aligned} \mu(s, \delta([|\det|^{-\alpha}\rho, &|\det|^{\alpha}\rho]) \otimes \tau) \\ =&\mu(s, \delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho]) \otimes \delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho]))^2 \\ &\times \mu(s, \delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho]) \otimes \tau_1). \end{aligned} \tag{5.4}$$

Since (see [Ze])

$$\delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho]) \times \delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho])$$

is irreducible, we see that

$$\mu(s, \delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho]) \otimes \delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho]))$$

has a double zero at $s=0$ (see [W2]). Now, combining (5.3) and (5.4) we see that

$$\mu(s, \delta([|\det|^{-\alpha}\rho, |\det|^{\alpha}\rho]) \otimes \sigma)$$

has a zero of order four at $s=0$. This contradicts [W2] and proves (i). Now, we prove (ii). First, we show that σ'' is in discrete series. If not, then we can find a supercuspidal unitary representation $\rho \in \operatorname{Irr} GL(m_\rho, F)$, $\alpha, \beta \in \mathbb{R}$, $\beta - \alpha \in \mathbb{Z}_{\geq 0}$ and $\alpha + \beta \leq 0$, such that

$$\sigma'' \hookrightarrow \delta([|\det|^{\alpha}\rho, |\det|^{\beta}\rho]) \rtimes \sigma_1, \tag{5.5}$$

for some irreducible representation σ_1. Then we obtain the following chain of equivariant maps (see [Ze]):

$$\begin{aligned} \sigma &\hookrightarrow \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]) \rtimes \sigma'' \\ &\hookrightarrow \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]) \times \delta([|\det|^{\alpha}\rho, |\det|^{\beta}\rho]) \rtimes \sigma_1 \\ &\to \delta([|\det|^{\alpha}\rho, |\det|^{\beta}\rho]) \times \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]) \rtimes \sigma_1. \end{aligned} \tag{5.6}$$

The composition of those 3 equivariant maps must be zero. (Otherwise, (5.6) will imply $\sigma \hookrightarrow \delta([|\det|^{\alpha}\rho, |\det|^{\beta}\rho]) \times \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]) \rtimes \sigma_1$ contradicting the square-integrability criterion for σ.) In particular, the last equivariant map is not an isomorphism. This implies that the segments $[|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]$ and $[|\det|^{\alpha}\rho, |\det|^{\beta}\rho]$ are linked (see [Ze]). Hence $\rho \simeq \chi_{G'}$, $\beta - l \in \mathbb{Z}$ and one of the following holds:

$$\begin{cases} \beta > l, -l+1 < \alpha \leq l+1 \\ -l \leq \beta < l, \alpha < -l+1. \end{cases}$$

The first case is not possible since $\alpha + \beta \leq 0$. Thus, σ embeds into the kernel of the last equivariant map:

$$\sigma \hookrightarrow \delta([|\ |^{\alpha}\chi_{G'}, |\ |^{l}\chi_{G'}]) \times \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{\beta}\chi_{G'}]) \rtimes \sigma_1.$$

Since

$$\alpha + l \leq -l + l = 0,$$

this contradicts the square-integrability criterion for σ. This completes proof that σ'' is in discrete series.

We prove the next claim in (ii). Assume that there is an irreducible representation σ_1 such that

$$\sigma'' \hookrightarrow |\ |^{l}\chi_{G'} \rtimes \sigma_1.$$

Note that σ_1 cannot be tempered. Otherwise, applying the assumption in (ii) we would obtain that σ and tempered subrepresentations of $\delta([|\ |^{-l}\chi_{G'}, |\ |^l\chi_{G'}]) \rtimes \sigma_1$ share the same supercuspidal support. This contradicts (i). Now, by the Langlands classification, we can find a supercuspidal unitary representation $\rho \in \operatorname{Irr} GL(m_\rho, F)$, $\alpha, \beta \in \mathbb{R}$, $\beta - \alpha \in \mathbb{Z}_{\geq 0}$ and $\alpha + \beta < 0$ such that

$$\sigma_1 \hookrightarrow \delta([|\det|^\alpha \rho, |\det|^\beta \rho]) \rtimes \sigma_2,$$

for some irreducible representation σ_2. We obtain the following chain of equivariant maps (see [Ze]):

$$\begin{aligned}\sigma'' &\hookrightarrow |\ |^l\chi_{G'} \rtimes \sigma_1 \hookrightarrow |\ |^l\chi_{G'} \times \delta([|\det|^\alpha \rho, |\det|^\beta \rho]) \rtimes \sigma_2 \\ &\to \delta([|\det|^\alpha \rho, |\det|^\beta \rho]) \times |\ |^l\chi_{G'} \rtimes \sigma_2.\end{aligned}$$

Arguing as in (5.6), since $\alpha < 0$ we conclude $\rho \simeq \chi_{G'}$, $\beta = l - 1$, and σ embeds into the kernel of the last equivariant map:

$$\sigma'' \hookrightarrow \delta([|\ |^\alpha\chi_{G'}, |\ |^l\chi_{G'}]) \rtimes \sigma_2.$$

Thus, by the square integrability criterion for σ'', we obtain

$$\alpha + l \in \mathbb{Z}_{>0} \tag{5.7}$$

On the other hand,

$$0 > \alpha + \beta = \alpha + l - 1,$$

implies that

$$\alpha + l < 1.$$

This contradicts (5.7).

It remains to prove the part of (ii) related to Jacquet modules. This has exactly the same proof as [M3, Theorem 2.3] where one should replace the first assumption in (B) with (equivalent) assumption that we have just proved. ∎

6. Main results

The first main result of this paper is the following

THEOREM 6.1: *Assume that $\sigma \in \operatorname{Irr} G_n$ ($n \geq n_0$) is a representation in discrete series. Let*

$$m_{temp}(\sigma) = \begin{cases} m(\sigma); & m(\sigma) > n + \eta_{G'} \\ n + \eta_{G'}; & m(\sigma) \leq n + \eta_{G'}. \end{cases}$$

Then we have the following:

(i) *If* m *satisfies* $m(\sigma) \leq m \leq m_{temp}(\sigma)$, *then all irreducible suquotients of* $\Theta(\sigma, m)$ *are tempered representations. More precisely, they are all in discrete series if one of the following holds:*

(1) $m < n + \eta_{G'}$

(2) $m = m(\sigma) = n + \eta_{G'}$

(3) $m = m(\sigma) > n + \eta_{G'}$ *and* σ *does not satisfy that*

$$\sigma \hookrightarrow \delta([| \ |^{n-m+\eta_{G'}+1}\chi_{G'}, | \ |^{m-n-\eta_{G'}}\chi_{G'}]) \rtimes \sigma'',$$

for some representation $\sigma'' \in \operatorname{Irr} G_{n''}$.

(ii) *If* $m(\sigma) < n + \eta_{G'}$, *then all irreducible subquotients* τ *of* $\Theta(\sigma, n + \eta_{G'})$ *are of the form*

$$\tau \hookrightarrow \chi_G \rtimes \tau_1,$$

where τ_1 *is an irreducible subquotient of* $\Theta(\sigma, n+\eta_{G'}-1)$ *in discrete series.*

(iii) *If* m *satisfies* $m > m_{temp}(\sigma)$, *then any irreducible quotient* $\sigma(m)$ *of* $\Theta(\sigma, m)$ *is a unique irreducible subrepresentation of*

$$| \ |^{n-m+\eta_{G'}}\chi_G \times | \ |^{n-m+\eta_{G'}+1}\chi_G \times \cdots \times | \ |^{n-m_{temp}(\sigma)-\eta_G}\chi_G \rtimes \sigma(m_{temp}(\sigma)),$$

for some irreducible quotient $\sigma(m_{temp}(\sigma))$ *of* $\Theta(\sigma, m_{temp}(\sigma))$. *All other irreducible subquotients of* $\Theta(\sigma, m)$ *are either tempered or of the form*

$$| \ |^{n-m+\eta_{G'}}\chi_G \times | \ |^{n-m+\eta_{G'}+1}\chi_G \times \cdots \times | \ |^{n-m_1-\eta_G}\chi_G \rtimes \sigma(m_1),$$

for some tempered irreducible subquotient $\sigma(m_1)$ *of* $\Theta(\sigma, m_1)$, *where* $m > m_1 \geq m_{temp}(\sigma)$.

Proof. Theorem 6.1 (i) follows at once from Theorem 4.2. We prove (ii). Since $m(\sigma) < n + \eta_{G'}$, we see $\Theta(\sigma, n + \eta_{G'} - 1) \neq 0$. Corollary 3.1 enables us to take an irreducible quotient τ_1 of $\Theta(\sigma, n + \eta_{G'} - 1)$. By (i), τ_1 is in discrete series. Applying, Theorem 2.2, $R_{P_1'}(\omega_{n,n+\eta_{G'}}$ has quotient $\chi_G \otimes \omega_{n,n+\eta_{G'}-1}$. Therefore, $\chi_G \otimes \sigma \otimes \tau_1$ is a quotient of $R_{P_1'}(\omega_{n,n+\eta_{G'}})$. Hence, the Frobenius reciprocity implies that there is a nonzero equivariant map $\omega_{n,n+\eta_{G'}} \to \sigma \otimes (\chi_G \rtimes \tau_1)$. In particular, $\Theta(\sigma, n + \eta_{G'})$ poses an irreducible subquotient that is not discrete series. Since all irreducible subquotients of $\Theta(\sigma, n + \eta_{G'})$ are tempered (see Theorem 4.2 (ii)) and they share a supercuspidal support (see Lemma 4.2), they must be of the form stated in (ii). This proves (ii). The part of (iii) about the quotients follows from (i), (ii), and Lemma 4.3 (ii) by an easy induction. The remainder of (ii) follows from Theorem 4.1 again by an easy induction. ∎

Now, assume that the residue characteristic of F is different from two. Then the Howe conjecture holds (see [W1]). More precisely, let $\sigma \in \operatorname{Irr} G_n$. Then $\Theta(\sigma, m)$ is zero or it has the unique maximal proper subrepresentation; the corresponding irreducible quotient we denote by $\sigma(m)$.

The next corollary generalizes [MVW, Théorème principal 1] from the case of σ is a supercuspidal representation to the case of a general representation in discrete series. It is originally proved using the conjectural classification of discrete series [Mœ, MT] in [M3] by an explicit determination of the lifts. (See [M3, Theorem 4.1]).

Corollary 6.1: *Assume that the residue characteristic of F is different from 2. Let $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$ be a representation in discrete series. Then there is a unique integer $m_{temp}(\sigma) \geq n+\eta_{G'}$ such that $\sigma(m)$ is tempered for $m(\sigma) \leq m \leq m_{temp}(\sigma)$. Moreover, $m > m_{temp}(\sigma)$ we have that $\sigma(m)$ is a unique irreducible (Langlands) subrepresentation of*

$$| \ |^{n-m+\eta_{G'}} \chi_G \times | \ |^{n-m+\eta_{G'}+1} \chi_G \times \cdots \times | \ |^{n-m_{temp}(\sigma)-\eta_G} \chi_G \rtimes \sigma(m_{temp}(\sigma)).$$

Proof. This follows directly from the previous theorem. ∎

The next theorem describes the structure of the full lift in important cases. Also, it settles a part of the conjecture made in the introduction of [M4].

Theorem 6.2: *Assume that the residue characteristic of F is different from 2. Let $\sigma \in \operatorname{Irr} G_n$ $(n \geq n_0)$ be a representation in discrete series. Then $\Theta(\sigma, m)$ is irreducible for $m(\sigma) \leq m \leq m_{temp}(\sigma)$. In particular, if $m \leq n + \eta_{G'}$, then $\Theta(\sigma, m)$ is irreducible or zero.*

Proof. First, assume that one of the following holds:

(1) $m < n + \eta_{G'}$
(2) $m = m(\sigma) = n + \eta_{G'}$
(3) $m = m(\sigma) > n + \eta_{G'}$ and all irreducible subquotients of $\Theta(\sigma, m)$ are in discrete series.

Then all irreducible subquotients of $\Theta(\sigma, m)$ are in discrete series. Since this representation is of finite length, it is in fact a tempered representation of finite length. Since G_n has finite center, discrete series are projective objects in the category of tempered representations of finite length ([W2], Corollaire III.7.2).

Hence, we obtain

$$\sigma(m) \hookrightarrow \Theta(\sigma, m)$$

by the definition of a projective object in an Abelian category. This implies $\sigma(m) \simeq \Theta(\sigma, m)$ proving the theorem in this case. According to Theorem 6.1 (i) and (ii) we need to consider two more cases. Assume that $m(\sigma) < n + \eta_{G'}$ and $m = n + \eta_{G'}$. Then all irreducible subquotients τ of $\Theta(\sigma, n + \eta_{G'})$ are of the form (see Theorem 6.1 (ii))

$$\tau \hookrightarrow \chi_G \rtimes \tau_1, \tag{6.1}$$

where τ_1 is an irreducible subquotient of $\Theta(\sigma, m-1)$ in discrete series. Applying what we have proved above, we obtain

$$\tau_1 \simeq \Theta(\sigma, m-1) \simeq \sigma(m-1).$$

Thus, (6.1) reads

$$\tau \hookrightarrow \chi_G \rtimes \sigma(m-1). \tag{6.2}$$

Applying the Frobenius reciprocity to (6.2), we obtain

$$R_{P_1'}(\tau) \twoheadrightarrow \chi_G \otimes \sigma(m-1).$$

Now, it is enough to show that $R_{P_1'}(\Theta(\sigma, m))$ contains $\chi_G \otimes \sigma(m-1)$ in its composition series with multiplicity one. We use the filtration of $R_{P_1'}(\omega_{n,m})$:

$$\begin{cases} J_{10} = \chi_G \otimes \omega_{n,m-1} & \text{(quotient)} \\ J_{11} = \operatorname{Ind}_{P_1 \times GL(1,F) \times G'_{m-1}}^{G_n \times GL(1,F) \times G_{m-1}}(\Sigma_1 \otimes \omega_{n-1,m-1}) & \text{(subrepresentation).} \end{cases}$$

As in the proof of Theorem 4.1 (see the part after (4.11)), we see that $\chi_G \otimes \sigma \otimes \sigma(m-1)$ is a subquotient of the image of J_{10} under the cannonical epimorphism $R_{P_1'}(\omega_{n,m}) \twoheadrightarrow \sigma \otimes R_{P_1'}(\Theta(\sigma, m))$. Hence it must be a subquotient of

$$\chi_G \otimes \sigma \otimes \Theta(\sigma, m-1) \simeq \chi_G \otimes \sigma \otimes \sigma(m-1),$$

completing the proof in this case.

It remains to consider the case $m = m(\sigma) > n + \eta_{G'}$ when $\Theta(\sigma, m)$ has a tempered, but not in discrete series, irreducible subquotient. Then (see Theorem 4.2 (iii))

$$\sigma \hookrightarrow \delta([|\ |^{-l+1}\chi_{G'}, |\ |^{l}\chi_{G'}]) \rtimes \sigma'', \tag{6.3}$$

for some representation $\sigma'' \in \operatorname{Irr} G_{n''}$ in discrete series, where $l = m - n - \eta_{G'}$.

We would like to apply ([M3], Lemma 5.2) to the embedding in (6.3). The proof of the embedding in Lemma 5.2 in [M3] is based only on Kudla's filtration of Jacquet modules and for that part the proof uses only the assumptions (i) and (iii) of Definition 5.1 in [M3]. (We check that (i) and (iii) hold in our case. Since $0 < l = m - n - \eta_{G'} = m'' - n'' - \eta_{G'}$, Lemma 4.3 (ii) implies that (iii) holds. Also, Theorem 5.1 (ii) implies that (i) holds.) Thus, we obtain

$$\sigma(m) \hookrightarrow \delta([|\ |^{-l+1}\chi_G, |\ |^{l}\chi_G]) \rtimes \sigma''(m''). \tag{6.4}$$

Also, Lemma 4.3 (ii) implies

$$\sigma''(m'') \hookrightarrow |\ |^{-l}\chi_G \rtimes \sigma''(m''-1). \tag{6.5}$$

Combining this with (6.4), we obtain the following chain of equivariant maps:

$$\begin{aligned}\sigma(m) \hookrightarrow \delta([|\ |^{-l+1}\chi_G, |\ |^{l}\chi_G]) \times |\ |^{-l}\chi_G \rtimes \sigma''(m''-1)\\ \to |\ |^{-l}\chi_G \times \delta([|\ |^{-l+1}\chi_G, |\ |^{l}\chi_G]) \rtimes \sigma''(m''-1).\end{aligned}$$

The composition of those equivariant maps must be zero, or otherwise

$$\sigma(m) \hookrightarrow |\ |^{-l}\chi_G \rtimes \tau',$$

for some irreducible representation τ'. Hence, by Lemma 4.3, $\Theta(\sigma, m-1) \neq 0$. This is a contradiction since $m = m(\sigma)$. Thus, we obtain

$$\sigma(m) \hookrightarrow \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \rtimes \sigma''(m''-1). \tag{6.6}$$

We claim

$$m(\sigma'') = m'' - 1. \tag{6.7}$$

If not, then $l - 1 \geq 0$ and Lemma 4.3 imply

$$\sigma''(m''-1) \hookrightarrow |\ |^{-l+1}\chi_G \rtimes \sigma''(m''-2).$$

Combining this with (6.6), we obtain the following chain of equivariant maps:

$$\begin{aligned}\sigma(m) \hookrightarrow \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \times |\ |^{-l+1}\chi_G \rtimes \sigma''(m''-2)\\ \simeq |\ |^{-l+1}\chi_G \times \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \rtimes \sigma''(m''-2).\end{aligned}$$

Hence

$$\sigma(m) \hookrightarrow |\ |^{-l+1}\chi_G \rtimes \tau',$$

for some irreducible representation τ'. Using Corollary 3.2 we obtain

$$\sigma \hookrightarrow |\ |^{-l+1}\chi_{G'} \rtimes \sigma_1,$$

for some irreducible representation σ_1, contradicting the square-integrability criterion for σ. This proves (6.7).

Since we already know (see Theorem 4.2 (iii)) that $\Theta(\sigma'', m''-1)$ contains an irreducible subquotient in discrete series, it must be irreducible by the already established part of Theorem 6.2: $\Theta(\sigma'', m''-1) \simeq \sigma''(m''-1)$. (We warn the reader that in the remainder of the proof we use the notation introduced in Theorem 4.2 (ii) and its proof.) The second displayed formula after (4.15) is valid here:

$$\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \rtimes \Theta(\sigma'', m'') \twoheadrightarrow \Theta(\sigma, m). \tag{6.8}$$

Next, the proof of Lemma 4.4 implies that the multiplicity of $\delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \otimes \sigma''(m''-1)$ in $\mu^*(\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \rtimes \Theta(\sigma'', m''))$ is exactly two. In particular, the multiplicity of $\delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \otimes \sigma''(m''-1)$ in $\mu^*(\Theta(\sigma, m))$ is at most two. Next, arguing as in the proof of Lemma 4.4, the multiplicity of the term $\delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \otimes \sigma''(m''-1)$ in $\mu^*(\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \rtimes \sigma''(m''))$ is exactly two. Since

$$\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \rtimes \Theta(\sigma'', m'') \twoheadrightarrow \delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \rtimes \sigma''(m'')$$

and since, by Lemma 4.2 and Theorems 4.1, 4.2 and 5.1 (i), all irreducible subquotients τ of $\Theta(\sigma, m)$ satisfy $\tau \hookrightarrow \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \rtimes \sigma''(m''-1)$ (in particular, $\mu^*(\tau) \geq \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \otimes \sigma''(m''-1)$), we conclude that (6.8) and computed multipilicities imply

$$\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \rtimes \sigma''(m'') \twoheadrightarrow \Theta(\sigma, m). \tag{6.9}$$

Next, Lemma 1.2, applied to the embedding in (6.5), implies

$$|\ |^{l}\chi_G \rtimes \sigma''(m''-1) \twoheadrightarrow \sigma''(m'').$$

Hence

$$\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \times |\ |^{l}\chi_G \rtimes \sigma''(m''-1) \twoheadrightarrow \Theta(\sigma, m). \tag{6.10}$$

Also, by [Ze], we obtain

$$\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \times |\ |^{l}\chi_G \rtimes \sigma''(m'') \twoheadrightarrow \delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \rtimes \sigma''(m''-1). \tag{6.11}$$

Again, as in Lemma 4.4, we see that the multiplicity of

$$\delta([|\ |^{-l}\chi_G, |\ |^{l}\chi_G]) \otimes \sigma''(m''-1)$$

in

$$\mu^*(\delta([|\ |^{-l}\chi_G, |\ |^{l-1}\chi_G]) \times |\ |^{l}\chi_G \rtimes \sigma''(m''-1))$$

and

$$\mu^*(\delta([|\ |^{-l}\chi_G, |\ |^l\chi_G]) \rtimes \sigma''(m''-1))$$

is exactly two. Thus, as before (see the proof of (6.9)), computed multipilicities, applied to (6.10) and (6.11) imply

$$\delta([|\ |^{-l}\chi_G, |\ |^l\chi_G]) \rtimes \sigma''(m''-1) \twoheadrightarrow \Theta(\sigma, m).$$

Since, the left-hand side is completely reducible, $\Theta(\sigma, m)$ is also completely reducible. Hence

$$\sigma(m) \simeq \Theta(\sigma, m).$$

This completes the proof of the theorem. ∎

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